SPECIAL CASES OF NAPOLEON TRIANGLES

by

Plarenta Bredehoft

An Abstract

of a thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science
in the Department of Mathematics and Computer Science
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The first chapter of this thesis provides a brief history of Napoleon’s theorem. Napoleon is attributed with proposing a theorem that bears his name is geometry. Since his authorship is often questioned, we examine: Could Napoleon have proposed the theorem that bears his name? Did he prove the theorem?

In the second chapter, special cases of Napoleon triangles are studied. The following questions are addressed: If an isosceles (right angle) mother triangle is given, what properties must the external and internal Napoleon triangles have? What properties must the external and internal Napoleon triangles of a mother triangle have to ensure that the mother triangle is an isosceles (right angle) triangle?

In the third chapter special cases of relative Napoleon triangles are studied. The same questions explored in Chapter 2 are asked here as well. If a triangle is an isosceles (right angle) mother triangle, what properties must the relative external and relative internal Napoleon triangles have? What properties must the relative external and relative internal Napoleon triangles of a mother triangle have to ensure that the mother triangle is an isosceles (right angle) triangle?
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Chapter 1

Some History of Napoleon’s Theorem

In this chapter, we offer some new evidence relevant to answering the question, "How is Napoleon’s theorem really related to Napoleon?” In mathematics, Napoleon’s theorem states that if equilateral triangles are drawn on the sides of any triangle, either all outward, or all inward, the centroids of those equilateral triangles are the vertices of an equilateral triangle. The triangles thus formed are called the outer Napoleon triangle and inner Napoleon triangle, respectively (figure 1.1). Moreover the difference in area of these two triangles equals the area of the original triangle.

The theorem in question is often attributed to Napoleon, however, not everyone agrees that Napoleon proposed and/or proved the theorem. Napoleon definitely could have proposed and proved this theorem. His name deserves to be associated with this theorem. My research found that Napoleon is the only one attributed in the old literature as the one that proposed the theorem to Lagrange for proof. Moreover while others have doubted Napoleon’s ability to prove the theorem, my work shows that he was very capable of proving it. Unfortunately there is no definite evidence that Napoleon did actually prove the theorem,
Figure 1.1: $\triangle ABC$ is the Mother triangle, $\triangle XYZ$ is the outer Napoleon triangle and $\triangle X'Y'Z'$ is the inner Napoleon triangle, first, last or not at all, therefore one can only speculate. When one considers Napoleon’s mathematical skills the chance that Napoleon actually solved this problem remains a real possibility.

Could Napoleon have proposed the theorem bearing his name? Examination of the evidence shows he definitely could have.

Napoleon Bonaparte was born on August 15, 1769 in Ajaccio, capital of Corsica. Corsica was briefly an independent Corsican Republic from 1755 until its conquest by France in 1769, the year that Napoleon was born. Corsica’s culture contains both French and Italian elements and its constitution while a Republic was written in Italian. The native Corsican language, an Italo-dalmatian language, was the official language in Corsica until 1859, and is recognized as a regional language by the French government. Napoleon became Emperor
of France from 1804 to 1814 and he died on 5 May 1821 in St. Helena, an English owned island in the South Atlantic ocean, after being caught and exiled there for the last six years of his life.

In 1797, Napoleon was elected to the Mechanics section of the National Institute of Sciences and Arts. Established in 1795, there were no honorary members. When he took his place in the Institute on his return from the army of Italy, he said he might consider himself as the tenth member in his class which consisted of about fifty. Several famous mathematicians made up his class. Langrange, Laplace, and Monge were at the head of this class [1]. It was rather a remarkable circumstance and one which attracted considerable notice at the time, to see the young General of the army of Italy take his place in the Institute, and publicly discuss profound metaphysical subjects with his colleagues. Napoleon was then called the Geometrician of battles, and the Mechanician of victory.

The first written evidence of Napoleon’s Theorem is found in a problem posed by Mr. W. Rutherford, of Woodburn and published in the ”New Mathematical Questions” in The Ladies’ Diary in 1825 [11]. Some scholars use this evidence to refute the claim that Napoleon was the first person to propose this theorem. In the later edition, in 1826, Mr. Tho. Burn, Mr. John Walker, Mr. Mason, Messrs, J. Baines, Tho, Hind-march and W.S.B. are credited for providing solutions to this problem [12]. The editor mentions that he regrettably left out several other elegant solutions, Mr. Isaac Brown’s being one of them, but there is no indication that Mr. W. Rutherford had provided a solution. Not everyone was credited for their contribution in the 1826 edition of The Ladies’ Diary. Mr. W. Rutherford’s question is widely known today as Napoleon’s theorem, though no mention of Napoleon is ever given in The Ladies’ Diary. The appearance of the result in The Ladies’ Diary was apparently
forgotten for many years.

Grübaum [10] pointed out that the earliest published work that he could find mentioning Napoleon’s name and this theorem was published in 1911. The 17th edition of Aureliano Faifofer’s, ”Elementi di geometria ad uso degli’ instituti tecnici (1° biennio) e dei licei”, contains the following on page 186 exercise number 494[8]: ”Teorema proposto per la dimostrazione da Napoleone a Lagrange.” (Translated: Theorem proposed for the proof from Napoleon to Lagrange). According to Grübaum, this is the earliest known published work mentioning Napoleon’s name in connection with the result. Grübaum did not know for sure if Napoleon was mentioned in any prior edition.

From the earlier editions I found the same information in the 15th edition as well [7]. As a matter of fact, it is very easy to conclude that the 17th edition is just a reprint of the 15th edition with a different cover. University of Michigan carries the hard copy of the 15th edition and through careful examination I noticed that the 17th edition, 1911[8], copied from Princeton University and available online at Hathi Trust digital Library, is exactly the same copy as the one in Michigan for the 15th edition, 1907. An electronic copy of the 15th edition, 1907, is available online as well, in google books, but I checked that the hard copy, available at University of Michigan, is actually identical to the one online. Until now the 17th edition, 1911 of Faifofer is referenced as the earliest proof that Napoleon had anything to do with this theorem. My research revealed that there is an earlier version published in 1907 that mentions the exact problem with Napoleon’s reference as well.

Some writers have raised the possibility that Faifofer’s publisher may have added the note about Napoleon since he was dead in 1911. This claim could be refuted, since Faifofer was still alive in 1907, when the 15th edition was published and he died in 1909.
Faifofer studied mathematics at the University of Padua, and was a teacher of geometry. He became the chair of mathematics at the high school Foscarini in Venice where he remained throughout his life. His books were translated into several languages. "Elementi di geometria ad uso degl’ instituti tecnici (1° biennio) e dei licei" written by Faifofer, was first published in 1878. Faifofer was born 22 years after Napoleon’s death and Lagrange died even before Napoleon did, in 1813. Since Faifofer was not their contemporary perhaps he could have read the fact mentioned in his book, that Napoleon proposed the theorem to Lagrange, from the writings of his time or from word of mouth.

Italian born, Faifofer was a mathematics professor. The purpose of his book, "Elementi di geometria ad uso degl’ instituti tecnici (1° biennio) e dei licei", was to teach students geometry and since it reached up to 17 editions it must have been very famous at the time. We do not know where or how he learned that Napoleon proposed this theorem to Italian born Joseph-Louis Lagrange (born Giuseppe Lodovico Lagrangia). Instead it appears that Faifofer simply stated that credit for proposing this problem is Napoleon’s. in my opinion for just proposing the problem, even though I am quite sure he was able to.

Nevertheless, Faifofer is not the only Italian writer to give credit to Napoleon. In the book "Un theorem di Napoleone", Rivista di Matem. Pura ed Applicata per gli student delle scule medie, 1 (1926) [4], The authors claim the result was presented for proof by Napoleon to the Italian mathematician Lagrange, which is exactly Faifofer’s claim as well. To completely dismiss these mathematicians’ statements that Napoleon proposed the theorem is a mistake. The evidence is credible. The fact that the same statement remains in later writings of Italian authors, in 1926 [4], suggests the evidence is credible. Several later writers wondered whether the result may have been known earlier. Davis states, "I conjectured that the whole
question was known in antiquity [6].”

Grünbaum [10] also mentions, ”according to Wetzel [17] , [the result] is surely one of the most-often rediscovered results in mathematics. Holmes states the result in 1874 as a fact without reference, and uses it in the proof of another result. Laisant [36, p. 148] mentions in 1877 the result as ”propriete bien connue”(Translated: property well known) [of triangles], without finding it necessary to give any specific reference.” Hence, by the late 1960’s, there have been several sources mentioning Napoleon’s theorem.

Napoleon was well known for posing mathematical puzzles, problems or questions to his staff. Napoleon took nearly 154 scientists to investigate Egypt’s history, geography and natural phenomena before becoming an emperor [16].

The Arcanum, published by John Bennet( the engineer), in 1838[2], includes on page 271, Napoleon’s problem to his staff and the figure demonstrating a solution provided by the author is right on the cover. Bennet writes the following history on the same page:

"The frontispiece to this work commences with the sublimely beautiful problem to his staff. The manner of obtaining this very valuable and desirable axiom is as follows:- During the publication of the work entitled, Geometrical Illustrations, on the 9th of May, 1836, a paper was left for the author thereof at the Publisher’s”. The following is a literal copy: viz.

"Napoleon on his voyage from Egypt, amused himself and staff with circular geometry; what circular geometry might be, was only to be collected from the tradition, that the problem given by the future Emperor was,To divide the circumference of a circle into four equal parts, by means of circles only. The story however created the impression, that the idea which had passed through the mind of that eminent practical geometer, was, that in the properties of a circle, or still more probably in the sphere, might be discovered the elements
of geometrical organization.”

Moreover, in 1797, Napoleon was discussing geometry with Joseph Louis Lagrange and Pierre Simon de Laplace. Napoleon surprised them by explaining some of Mascheroni’s solutions that were completely new to them about compass geometry. Laplace reportedly remarked: ”We expect all things from you, General, except a lesson in geometry” (Laplace, 1797). ‘Nous attendions tout de vous, general, excepte des lecons de Mathematiques’ [14].

Whether this is true or not, Napoleon did introduce Mascheroni’s compass work to French mathematicians. A translation of ”Geometria del Compasso” was published in Paris in 1798, one year after the Italian edition of this book was published, in Italy.”[15]

So since Napoleon was called ”eminent practical geometer” and posed questions to his staff, giving lessons in geometry, even to Laplace, it is not hard to imagine him posing questions to his close mathematician friends and scientists, including Fourier, Monge, Laplace, Chaptal, Lagrange and Berthollet. So indeed ”Teorema proposto per la dimostrazione da Napoleone a Lagrange” might have very well been one problem that he would have posed for proof to Lagrange.

Could he have proved it?

Dr. E. Andrew Boyd, also explored the question whether Napoleon actually did prove the theorem[3]. Dr. Boyd pointed out that a lot of fuel for the debate was provided by an off-hand comment of two twentieth century mathematicians. One was the famed geometer Donald Coxeter. In the textbook, Geometry Revisited, he and co-author Samuel Greitzer write, ”the possibility of Napoleon knowing enough geometry [to prove the result] is as questionable as the possibility that he knew enough English to compose the famous palindrome ABLE WAS I ERE I SAW ELBA” [5]. Coxeter and Greitzer didn’t just challenge the claim that Napoleon
was first to discover the theorem. They didn’t think Napoleon was capable of solving it at all. Dr. E. Andrew Boyd suggests "It’s quite an insult coming from the English born Coxeter”.

So could Napoleon have proved the result? Dr. Boyd writes, "A review of the many proofs leaves little doubt he certainly could have. Napoleon’s Theorem requires logical thinking but little more. Most proofs of it are understandable by a good high school student. What led Coxeter and Greitzer to disparage Napoleon’s abilities isn’t clear, though it may have been just a poor effort at humor” [3]. When one considers the additional evidence of Napoleon studying, teaching, discussing and applying mathematics, Boyd’s conclusion that Napoleon could have proved the theorem is sound.

But was Napoleon the first to discover the result that bears his name and should his name be associated with the theorem?

Napoleon, according to National Galleria of Victoria [16], 2 June 7 October 2012 edition, is known for reforming public education based on the ideals of reason and use of intelligent debate to build an ordered society. He also consolidated a system of primary, secondary, and technical schools and universities, regulated by the State with centrally recruited teachers. Education in the sciences was made a cornerstone of the curriculum.

He was perhaps the most important single cause of nationalistic movements that spread through Europe during the nineteenth century. On the same library shelf, one may find scholarly books praising him as one of the greatest and most admirable men in history alongside equally learned volumes blasting him as one of the worst calamities ever to afflict the world. All these things have come to overshadow that fact the Napoleon was also a mathematician.

Napoleon discussed mathematical principles with his fellow mathematicians, posed ques-
tions to his staff and was a well-known geometer of his time. So could Napoleon have proposed the theorem that bears his name? Definitely. Could he have proved the theorem that bears his name? Most definitely. Did he prove the theorem? We don’t know. In my opinion, since nobody else has claimed this problem as his own in written literature and Napoleon is attributed with proposing the theorem, he deserves the name on the theorem more than anyone. Was Napoleon the first to discover the result that bears his name? Probably not, as history repeats itself and people keep rediscovering results. Does he deserve to have his name associated with this theorem? Definitely.
Chapter 2

Napoleon Triangles

2.1 Internal and External Napoleon $\triangle$s

Definition 2.1.1. Let $\triangle ABC$ denote any triangle. Then the triangle whose vertices are given by the points of the centroids of the external equilateral triangles with their basis on $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$ is called the external Napoleon triangle, denoted by $\triangle e$ (figure 2.1).

Figure 2.1: $\triangle ABC$ is the Mother triangle and $\triangle XYZ$ is the external Napoleon triangle
**Definition 2.1.2.** Let $\triangle ABC$ denote any triangle. Then the triangle whose vertices are given by the points of the centroids of the internal equilateral triangles with their bases on $AB$, $BC$, and $AC$ is called the *internal Napoleon triangle*, denoted by $\triangle i$ (figure 2.2).

![Diagram](image)

**Figure 2.2:** $\triangle ABC$ is the Mother triangle and $\triangle X'Y'Z'$ is the internal Napoleon triangle

In this thesis, the original triangle $\triangle ABC$ is often referred as the *mother triangle* of its internal and external Napoleon triangles.

Napoleon’s theorem states that if equilateral triangles are drawn on the sides of any triangle, either all outward, or all inward, the centroids of those equilateral triangles are the vertices of an equilateral triangle.

The triangle thus formed is called the Napoleon triangle (inner and outer). Moreover the difference in area of these two triangles equals the area of the original triangle.

The following lemma follows from the results in [9].

**Lemma 2.1.3.** Let $\triangle XYZ$ and $\triangle X'Y'Z'$ be the $\triangle e$ and $\triangle i$ of their mother triangle $\triangle ABC$, respectively, as shown in figure 1.2. If $\triangle ABC$ is an isosceles triangle such that $A =$
\((-b, 0), B = (b, 0), \) and \(C = (0, c)\) where \(b, c > 0\), then \(X = \left(\frac{b}{2} + \frac{\sqrt{3}c}{6}, \frac{\sqrt{3}b}{6} + \frac{c}{2}\right), \ Y = \left(-\frac{b}{2} - \frac{\sqrt{3}c}{6}, \frac{\sqrt{3}b}{6} + \frac{c}{2}\right), \ X' = \left(\frac{b}{2} - \frac{\sqrt{3}c}{6}, -\frac{\sqrt{3}b}{6} + \frac{c}{2}\right), \ Y' = \left(-\frac{b}{2} + \frac{\sqrt{3}c}{6}, -\frac{\sqrt{3}b}{6} + \frac{c}{2}\right), \ Z = \left(0, -\frac{\sqrt{3} \cdot b}{3}\right)\) and \(Z' = \left(0, \frac{\sqrt{3} \cdot b}{3}\right)\).

The following two lemmas are Theorems 2 and 3 of [13].

**Lemma 2.1.4.** The centroids of the external Napoleon triangle and its mother triangle are coincident.

**Lemma 2.1.5.** The centroids of the internal Napoleon triangle and its mother triangle are coincident.

### 2.2 The \(\triangle i\) and \(\triangle e\) of an isosceles \(\triangle\)

**Theorem 2.2.1.** Let \(\triangle XYZ\) and \(\triangle X'Y'Z'\) be the \(\triangle e\) and \(\triangle i\) of their mother triangle \(\triangle ABC\), respectively. Suppose \(\overline{AB}\) is the shortest side in \(\triangle ABC\) and \(M\) is the centroid of \(\triangle ABC\) (figure 2.3). If \(\triangle ABC\) is an isosceles triangle with \(\overline{AC} \cong \overline{BC}\), then

1. the lines \(\overline{XY}, \overline{X'Y'},\) and \(\overline{AB}\) are parallel to each other;

2. the length of the base, \(|\overline{AB}|\) of \(\triangle ABC\) is equal to the difference of the lengths of the sides of \(\triangle e\) to \(\triangle i\), \(|\overline{XY}| - |\overline{X'Y'}|\);

3. the line segments \(\overline{XX'}\) and \(\overline{CB}\) are perpendicular bisectors of each other, and the line segments \(\overline{YY'}\) and \(\overline{CA}\) are perpendicular bisectors of each other;
4. the lines $\overrightarrow{CM}$, $\overrightarrow{XX'}$, and $\overrightarrow{YY'}$ are concurrent; and

5. the points $C$, $M$, $Z'$, and $Z$ are collinear.

Figure 2.3: $\triangle ABC$ is an isosceles triangle, $|AB| < |BC|$

Proof. 1. Since $\triangle ABC$ is an isosceles triangle with $\overrightarrow{AC} \equiv \overrightarrow{BC}$, without loss of generality let $A = (-b, 0)$, $B = (b, 0)$, and $C = (0, c)$ with $b, c > 0$. It follows from Lemma 2.1.3 that the slopes of $\overrightarrow{AB}$, $\overrightarrow{XY}$, and $\overrightarrow{X'Y'}$ are zero, and hence $\overrightarrow{AB} \parallel \overrightarrow{XY} \parallel \overrightarrow{X'Y'}$. 

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Moreover, since both $\triangle XYZ$ and $\triangle X'Y'Z'$ are equilateral triangles and $XY \parallel X'Y'$, the other two corresponding sides are parallel as well.

2. Since $X = \left( \frac{b}{2} + \frac{\sqrt{3}c}{6}, \frac{\sqrt{3}b}{6} + \frac{c}{2} \right)$, $Y = \left( -\frac{b}{2} - \frac{\sqrt{3}c}{6}, \frac{\sqrt{3}b}{6} + \frac{c}{2} \right)$, $X$ and $Y$ are symmetric with respect to $\overrightarrow{CM}$. Similarly, $X'$ and $Y'$ are symmetric with respect to $\overrightarrow{CM}$. Moreover the difference of the $x$-coordinates of $X$ and $Y'$ is $\left( \frac{b}{2} + \frac{\sqrt{3}c}{6} \right) - \left( -\frac{b}{2} + \frac{\sqrt{3}c}{6} \right) = b$. It follows that

$$\overline{AB} = |XY| - |X'Y'|.$$

3. Since the midpoints of $XX'$ and $CB$ are $\left( \frac{b}{2}, \frac{c}{2} \right)$ and the product of the slopes of $\overrightarrow{XX'}$ and $\overrightarrow{CB}$ is equal to $-1$, $XX'$ and $CB$ are perpendicular bisectors of each other. Similarly, $YY'$ and $CA$ are perpendicular bisectors of each other.

4. Solving the system

$$y - \frac{c}{2} = \frac{b}{c} \left( x - \frac{b}{2} \right),$$

$$x = 0,$$

the intersecting point of $\overrightarrow{XX'}$ and $\overrightarrow{CM}$ is $\left( 0, \frac{c^2 - b^2}{2c} \right)$. Similarly, the intersecting point of $\overrightarrow{YY'}$ and $\overrightarrow{CM}$ is $\left( 0, \frac{c^2 - b^2}{2c} \right)$. Therefore, the lines $\overrightarrow{CM}$, $\overrightarrow{XX'}$, and $\overrightarrow{YY'}$ are concurrent.

5. The points $C$, $M$, $Z'$, and $Z$ are collinear on the vertical line $x = 0$. 

\[\square\]
Theorem 2.2.2. Let $\triangle XYZ$ and $\triangle X'Y'Z'$ be the Napoleon $\triangle e$ and Napoleon $\triangle i$ of their mother isosceles triangle $\triangle ABC$, with $AC \cong BC$, respectively (figure 2.4). If $AB$ is the longest side in $\triangle ABC$, then $|AB| = |XY| + |X'Y'|$.

Figure 2.4: $\triangle ABC$ is an isosceles triangle, $|AB| > |BC|$

Proof. Note that the $x$-coordinate of $X$ is $b/2 + \sqrt{3}/6 \ c$ and the $x$-coordinate of $Y$ is $b/2 - \sqrt{3}/6 \ c$. So $X$ and $Y$ are symmetric with respect to $\overrightarrow{CM}$. Similarly, $X'$ and $Y'$ are symmetric with respect to $\overrightarrow{CM}$, where $M$ is the centroid of $\triangle ABC$. Moreover the sum of the $x$-coordinates of $X$ and $Y'$ is $(b/2 + \sqrt{3}/6 \ c) + (b/2 - \sqrt{3}/6 \ c) = b$. It follows that $|AB| = |XY| + |X'Y'|$. 

$\square$
The next theorem follows from Theorems 1.2.1. and 1.2.2., and hence the proof is omitted.

**Theorem 2.2.3.** If \( \triangle ABC \) is an equilateral triangle with centroid \( M \), and \( \triangle X'Y'Z' \) and \( \triangle XYZ \) are its internal Napoleon and external Napoleon triangles, respectively, then \( \triangle ABC \cong \triangle XYZ \) and \( \triangle X'Y'Z' \) is the degenerate triangle \( M \) (figure 2.5).

![Figure 2.5: \( \triangle ABC \) is an equilateral triangle](image)

**Theorem 2.2.4.** Let \( \triangle XYZ \) and \( \triangle X'Y'Z' \) be the \( \Delta e \) and \( \Delta i \) of their mother triangle \( \triangle ABC \), respectively. Suppose \( \overline{AB} \) is the shortest side and \( M \) is the centroid of \( \triangle ABC \) (figure 2.6). If

1. the lines \( \overrightarrow{XY}, \overrightarrow{X'Y'}, \) and \( \overrightarrow{AB} \) are parallel to each other;

2. the line segments \( \overline{XX'} \) and \( \overline{CB} \) are perpendicular bisectors of each other, and the line segments \( \overline{YY'} \) and \( \overline{CA} \) are perpendicular bisectors of each other;

3. the lines \( \overrightarrow{CM}, \overrightarrow{XX'}, \) and \( \overrightarrow{YY'} \) are concurrent; and
4. the points $C$, $M$, $Z'$, and $Z$ are collinear,

then $\triangle ABC$ is an isosceles triangle with $AC \cong BC$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.6.png}
\caption{$\triangle ABC$ is an isosceles triangle}
\end{figure}

\textit{Proof.} Let $XX' \cap BC = \{V\}$, $YY' \cap AC = \{W\}$, and $\overrightarrow{XX'}$, $\overrightarrow{YY'}$, and $\overrightarrow{CM}$ meet at $U$. Since $M$ is the centroid of the equilateral $\triangle XYZ$, $\triangle XMY$ is an isosceles triangle, and hence $\overline{MX} \cong \overline{MY}$. Similarly, $\triangle X'MY'$ is an isosceles triangle, and hence $\overline{MX'} \cong \overline{MY'}$. It follows that

$$|XY'| = |XM| - |MY'| = |YM| - |MX'| = |YX'|.$$

Since $XY \cong Y'X$, $\angle X'YX \cong \angle Y'XY$, and $YY' \cong YY'$, $\triangle XYY \cong \triangle Y'Y'X$. Therefore,
\(XX' \cong YY',\) with

\[
|XX'| = \frac{1}{2}|XX| = \frac{1}{2}|YY'| = |YW|.
\]

Since \(\triangle XX'Y \cong \triangle YY'X,\)

\[
\angle X'XY = \angle UX'Y \cong \angle UYX = \angle Y'YX,
\]

and hence \(UX \cong UY.\) Therefore,

\[
|UV| = |UX| - |XV| = |UY| - |YW| = |UW|.
\]

Let \(CZ \cap XY = \{N\}.\) Then \(\triangle YUN \cong \triangle XUN\) since \(\overrightarrow{YX} \perp \overrightarrow{CZ},\ YU \cong Xu, \angle YNU = 90^\circ = \angle XNU,\) and \(NU \cong NU.\) Therefore,

\[
\angle CUY = \angle NUY \cong \angle NUX = \angle CUX.
\]

Since \(\overrightarrow{WU} \cong \overrightarrow{WU}, \angle CUV \cong \angle CUV, \overrightarrow{CU} \cong \overrightarrow{CU}, \triangle CWF \cong \triangle CVU.\) Therefore, \(\overrightarrow{CW} \cong \overrightarrow{CV}.\) It follows that

\[
|AC| = 2|CW| = 2|CV| = |AB|.
\]

2.3 \(\triangle i\) and \(\triangle e\) of a right \(\triangle\)

The following lemma follows from the results in [9].

**Lemma 2.3.1.** Let \(A = (0, 0), B = (b, 0)\) and \(C = (0, c)\) with \(b, c > 0\) be the vertices of a right triangle and \(\triangle DEP\) and \(\triangle D'E'P'\) be its Napoleon’s \(\triangle e\) and Napoleon \(\triangle i,\) respectively. Then \(D = \left(\frac{3b + \sqrt{3}c}{6}, \frac{3c + \sqrt{3}b}{6}\right), E = \left(-\frac{\sqrt{3}c}{6}, \frac{c}{2}\right), P = \left(\frac{b}{2}, -\frac{\sqrt{3}b}{6}\right),\)

\(D' = \left(\frac{3b - \sqrt{3}c}{6}, \frac{3c - \sqrt{3}b}{6}\right), E' = \left(\frac{\sqrt{3}c}{6}, \frac{c}{2}\right),\) and \(P' = \left(\frac{b}{2}, \frac{\sqrt{3}b}{6}\right).\)
Theorem 2.3.2. Let $\triangle D'E'P'$ and $\triangle D'E'P'$ be the $\triangle e$ and $\triangle i$ of the mother triangle $\triangle ABC$, respectively. Then $\triangle ABC$ is a right triangle with the right angle at $A$, see figure 2.7, if and only if

1. the line segments $\overline{DD'}$ and $\overline{BC}$ are perpendicular bisectors of each other, the line segments $\overline{EE'}$ and $\overline{AC}$ are perpendicular bisectors of each other, and the line segments $\overline{PP'}$ and $\overline{AB}$ are perpendicular bisectors of each other;

2. the lines $\overrightarrow{DD'}$, $\overrightarrow{EE'}$, and $\overrightarrow{PP'}$ meet at the midpoint of $\overline{BC}$; and

3. the lines $\overrightarrow{EE'}$ and $\overrightarrow{PP'}$ are perpendicular.

Figure 2.7: $\triangle ABC$ is a right triangle
Proof. ($\Rightarrow$)

1. Assume that $\angle BAC = 90^\circ$ and $\triangle ABC$ is the mother triangle. It follows from Lemma 2.3.1 that the midpoint $L$ of $DD'$ is $\left(\frac{b}{2}, \frac{c}{2}\right)$; the midpoint $O$ of $EE'$ is $\left(0, \frac{c}{2}\right)$; and the midpoint $N$ of $PP'$ is $\left(\frac{b}{2}, 0\right)$. It is clear that the slopes of the lines $\overrightarrow{DD'}$ and $\overrightarrow{CB}$ are negative reciprocals of each other; the lines $\overrightarrow{EE'}$ and $\overrightarrow{AC}$ are horizontal and vertical lines, respectively; and the lines $\overrightarrow{PP'}$ and $\overrightarrow{AB}$ are vertical and horizontal lines, respectively. Therefore, the line segments $\overline{DD'}$ and $\overline{BC}$ are perpendicular bisectors of each other; the line segments $\overline{EE'}$ and $\overline{AC}$ are perpendicular bisectors of each other; and the line segments $\overline{PP'}$ and $\overline{AB}$ are perpendicular bisectors of each other.

2. Since $L = \left(\frac{b}{2}, \frac{c}{2}\right)$ satisfies the equations of $\overrightarrow{DD'}$, $\overrightarrow{EE'}$, and $\overrightarrow{PP'}$, these lines meet at the midpoint of $\overline{BC}$.

3. Since $O$ and $L$ are midpoints of $\overline{AC}$ and $\overline{BC}$, $\overline{OL} \parallel \overline{AB}$, and hence the lines $\overrightarrow{EE'}$ and $\overrightarrow{PP'}$ are perpendicular.

($\Leftarrow$) Since $\overrightarrow{CA}$ and $\overrightarrow{EE'}$ are perpendicular bisectors of each other and $\overline{CA} \cap \overline{EE'} = \{O\}$, $\angle COE' = 90^\circ$. Similarly, $\overrightarrow{DD'}$ and $\overrightarrow{CB}$ are perpendicular bisectors of each other. Since $O$ and $L$ are midpoints of $\overline{AC}$ and $\overline{BC}$, respectively, $\overline{OL} \parallel \overline{AB}$, and hence $\angle BAC = \angle LOC = 90^\circ$. \hfill $\square$
Chapter 3

Relative Napoleon triangles

3.1 Relative Internal and External Napoleon △s

Suppose an arbitrary △ABC is given. We follow similar steps as in Napoleon’s Theorem, i.e., on each of the sides \( \overline{AB} \), \( \overline{BC} \), \( \overline{CA} \) construct the exterior equilateral triangles \( AC'B \), \( BA'C \), and \( CB'A \); and the interior equilateral triangles, \( AC''B \), \( BA''C \), and \( CB''A \). The next step is different. Let \( C_1 \) be the midpoint of \( A'B' \), \( A_1 \) the midpoint of \( B'C' \), \( B_1 \) the midpoint of \( A'C' \) (figure 3.1). Then (1) \( \triangle A_1B_1C_1 \), \( \triangle B_1C_1A_1 \), and \( \triangle C_1A_1B_1 \) are equilateral triangles with the same orientation as triangle \( \triangle ABC \), as shown in Grunbaum’s article [9].

(2) Let \( C^* \), \( A^* \), \( B^* \) be the centroid of \( \triangle A_1B_1C_1 \), \( \triangle B_1C_1A_1 \), and \( \triangle C_1A_1B_1 \), respectively. Then \( \triangle A^*B^*C^* \) is an equilateral triangle [9], we call this the relative exterior Napoleon triangle or relative Napoleon △e.

Similarly, for the interior equilateral triangles \( AC''B \), \( BA''C \), and \( CB''A \), let \( C_2 \) be the midpoint of \( A''B'' \), \( A_2 \) the midpoint of \( B''C'' \), \( B_2 \) the midpoint of \( A''C'' \). Then (1) \( \triangle A_2B_2C_2 \), \( \triangle B_2C_2A_2 \), and \( \triangle C_2A_2B \) are equilateral triangles with orientation opposite to that
Figure 3.1: $\triangle ABC$ is scalene

of $\triangle ABC$. (2) Let $A^{**}$, $B^{**}$, $C^{**}$ be the centroids of triangles $\triangle A_2B_2C$, $\triangle B_2C_2A$, and $\triangle C_2A_2B$, respectively, as mentioned in Grunbaum’s article [9]. Then $\triangle A^{**}B^{**}C^{**}$ is an equilateral triangle, as shown in [9], we call this the relative interior Napoleon triangle or relative Napoleon $\triangle i$.

We ask the same questions as in Chapter 1. If $\triangle ABC$ is an isosceles triangle, what properties must triangles $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$ have? What properties must triangles $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$ have to ensure that $\triangle ABC$ is an isosceles triangle? Similarly, if $\triangle ABC$ is a right triangle, what properties must triangles $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$ have?
What properties must triangles $\triangle A'B'C'$ and $\triangle A''B''C''$ have to ensure that $\triangle ABC$ is a right triangle?

### 3.2 Relative Napoleon $\triangle i$ and $\triangle e$ of an isosceles $\triangle$

In this section, we answer the following questions: If $\triangle ABC$ is an isosceles triangle, what properties must triangles $\triangle A'B'C'$ and $\triangle A''B''C''$ have? What properties must triangles $\triangle A'B'C'$ and $\triangle A''B''C''$ have to ensure that $\triangle ABC$ is an isosceles triangle?

**Lemma 3.2.1.** Let $\triangle A'B'C'$ and $\triangle A''B''C''$ be the relative Napoleon $\triangle e$ and relative Napoleon $\triangle i$ of a given triangle $\triangle ABC$. If $\triangle ABC$ is isosceles with $AC \cong BC$, then the centroids of $\triangle ABC$, $\triangle A'B'C'$ and $\triangle A''B''C''$ coincide.

**Proof.** Without loss of generality, let $A = (-b, 0)$, $B = (b, 0)$, and $C = (0, c)$ with $b, c > 0$, and $G$, $G^*$ and $G^{**}$ be the centroid of $\triangle ABC$, $\triangle A'B'C'$ and $\triangle A''B''C''$, respectively.

Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$, Grunbaum [9] shows that

\[
A^* = \left( \frac{6a_1 + 3b_1 + 3c_1 + \sqrt{3}(b_2 - c_2)}{12}, \frac{6a_2 + 3b_2 + 3c_2 + \sqrt{3}(b_1 - c_1)}{12} \right)
\]

\[
B^* = \left( \frac{6b_1 + 3c_1 + 3a_1 + \sqrt{3}(a_2 - c_2)}{12}, \frac{6b_2 + 3a_2 + 3c_2 + \sqrt{3}(c_1 - a_1)}{12} \right)
\]

and

\[
C^* = \left( \frac{6c_1 + 3b_1 + 3a_1 + \sqrt{3}(b_2 - a_2)}{12}, \frac{6c_2 + 3a_2 + 3b_2 + \sqrt{3}(a_1 - b_1)}{12} \right)
\]

Similarly,

\[
A^{**} = \left( \frac{6a_1 + 3b_1 + 3c_1 + \sqrt{3}(b_2 - c_2)}{12}, \frac{6a_2 + 3b_2 + 3c_2 + \sqrt{3}(b_1 - c_1)}{12} \right)
\]

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\[ B^{**} = \left( \frac{6b_1 + 3c_1 + 3a_1 + \sqrt{3}(c_2 - a_2)}{12}, \frac{6b_2 + 3a_2 + 3c_2 + \sqrt{3}(-c_1 + a_1)}{12} \right) \]
and
\[ C^{**} = \left( \frac{6c_1 + 3b_1 + 3a_1 + \sqrt{3}(-b_2 + a_2)}{12}, \frac{6c_2 + 3a_2 + 3b_2 + \sqrt{3}(-a_1 + b_1)}{12} \right) \]

For our case \( A = (-b, 0) \), \( B = (b, 0) \), and \( C = (0, c) \), above coordinates are:

\[ A^* = \left( \frac{-3b + \sqrt{3}c}{12}, \frac{3c + \sqrt{3}b}{12} \right) \]
\[ B^* = \left( \frac{3b - \sqrt{3}c}{12}, \frac{3c + \sqrt{3}b}{12} \right) \]
and
\[ C^* = \left( 0, \frac{6c - 2\sqrt{3}b}{12} \right) \]

Similarly,

\[ A^{**} = \left( \frac{-3b - \sqrt{3}c}{12}, \frac{3c - \sqrt{3}b}{12} \right) \]
\[ B^{**} = \left( \frac{3b + \sqrt{3}c}{12}, \frac{3c - \sqrt{3}b}{12} \right) \]
and
\[ C^{**} = \left( 0, \frac{6c + 2\sqrt{3}b}{12} \right) \]

First, find the midpoint \( L^* \) of \( A^*B^* \). Then the x and y coordinates of \( G^* \) are found by using the formulas:

\[ x_{G^*} = \left( \frac{x_{L^*} + rx_{C^*}}{1 + r} \right) \]
and
\[ y_{G^*} = \left( \frac{y_{L^*} + ry_{C^*}}{1 + r} \right) \]
where $r = 1/2$.

Similarly, we find the midpoint $L^{**}$ of $A^{**}B^{**}$. Then the $x$ and $y$ coordinates of $G^{**}$ are found by using the formulas:

$$x_{G^{**}} = \left( \frac{x_{L^{**}} + rx_{C^{**}}}{1 + r} \right)$$

and

$$y_{G^{**}} = \left( \frac{y_{L^{**}} + ry_{C^{**}}}{1 + r} \right)$$

where $r = 1/2$. Then

$$G = G^* = \left( 0, \frac{c}{3} \right) = G^{**}.$$

\[ \square \]

**Theorem 3.2.2.** Let $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$ be relative Napoleon $\triangle e$ and relative Napoleon $\triangle i$ of a given triangle $\triangle ABC$. Then $\triangle ABC$ is isosceles with $AC \cong BC$, see figure 3.2, if and only if

1. If $G, G^*$ and $G^{**}$ are the centroids of $\triangle ABC$, $\triangle A^*B^*C^*$ and $\triangle A^{**}B^{**}C^{**}$, respectively, then $G$, $G^*$ and $G^{**}$ coincide.

2. the points $C$, $C^*$, $C^{**}$, and $G$ are collinear.

3. the lines $\overrightarrow{A^*B^*}$ and $\overrightarrow{A^{**}B^{**}}$ are parallel to $\overrightarrow{AB}$;

**Proof.** ($\Rightarrow$) Since $\triangle ABC$ is isosceles with $AC \cong BC$, without loss of generality, let $A = (-b, 0)$, $B = (b, 0)$, and $C = (0, c)$.

1. See Lemma 3.2.1.
Figure 3.2: $\triangle ABC$ is an isosceles triangle, $|\overline{AB}| < |\overline{BC}|$

2. Since $C = (0, c)$, $C^* = \left(0, \frac{6c - 2\sqrt{3}b}{12}\right)$, $C^{**} = \left(0, \frac{6c + 2\sqrt{3}b}{12}\right)$ and $G = \left(0, \frac{c}{3}\right)$, they lie on the vertical line $x = 0$.

3. Since the y-coordinates of $A^*$ and $B^*$ are the same and the y-coordinates of $A^{**}$ and $B^{**}$ are the same, the statement is true since the slopes of these lines equal to 0.

$(\Leftarrow)$ Since the points $C$, $C^*$, $C^{**}$ and $G$ are collinear and $\triangle A^{**}B^{**}C^{**}$ is an equilateral triangle, $\overrightarrow{CC^{**}}$ is the perpendicular bisector of $\overline{A^{**}B^{**}}$, and hence $\overline{A^{**}C} \cong \overline{B^{**}C}$. Let $P$ and $Q$ be the intersecting points of $\overrightarrow{A^{**}B^{**}}$ with $\overline{AC}$ and $\overline{BC}$, respectively. Since $\angle ACA^{**} \cong \angle BCB^{**}$, $\triangle A^{**}PC \cong \triangle B^{**}QC$. Since $\overrightarrow{A^{**}B^{**}} \parallel \overrightarrow{AB}$, $\angle BAC \cong \angle A^{**}PC \cong \angle B^{**}QC \cong \angle ABC$, and hence $\overline{AC} \cong \overline{BC}$. □
3.3 Relative Napoleon $\triangle i$ and $\triangle e$ of a right $\triangle$

In this section, we answer the questions: If $\triangle ABC$ is a right triangle, what properties must triangles $\triangle A'B'C'$ and $\triangle A''B''C''$ have? Conversely, what properties must triangles $\triangle A'B'C'$ and $\triangle A''B''C''$ have to ensure that $\triangle ABC$ is a right triangle?

**Theorem 3.3.1.** Let $\triangle A'B'C'$ and $\triangle A''B''C''$ be the relative Napoleon $\triangle e$ and Relative Napoleon $\triangle i$ of $\triangle ABC$, respectively. Let $O$ be the intersecting point of the lines $\overrightarrow{C''C}$ and $\overrightarrow{B''B}$. The mother triangle $\triangle ABC$ is a right triangle with the right angle at vertex $A$, see figure 3.3, if and only if

1. The lines $\overrightarrow{A'A}$, $\overrightarrow{B'B}$, and $\overrightarrow{C'C}$ are concurrent at the point $O$;

2. $\overrightarrow{A'A} \perp \overrightarrow{BC}$, $\overrightarrow{B'B} \perp \overrightarrow{AC}$, and $\overrightarrow{C'C} \perp \overrightarrow{AB}$; and

3. $\angle B''OC''$ is a right triangle.

**Proof.** $(\Rightarrow)$ Let $A = (0, 0)$, $B = (b, 0)$ and $(0, c)$ with $b, c > 0$ be the vertices of the right triangle with its right angle at $A$.

1. From Grunbaum’s article [9], $A^* = \left(\frac{3b + \sqrt{3}c}{12}, \frac{3c + \sqrt{3}b}{12}\right)$, $B^* = \left(\frac{6b - \sqrt{3}c}{12}, \frac{c}{4}\right)$, $C^* = \left(\frac{b}{4}, \frac{6c - \sqrt{3}b}{12}\right)$, $A^{**} = \left(\frac{3b - \sqrt{3}c}{12}, \frac{3c - \sqrt{3}b}{12}\right)$, $B^{**} = \left(\frac{6b + \sqrt{3}c}{12}, \frac{c}{4}\right)$, and $C^{**} = \left(\frac{b}{4}, \frac{6c + \sqrt{3}b}{12}\right)$. Solving the equations of the lines $\overrightarrow{B''B^{**}}$ and $\overrightarrow{C^{**}C^{**}}$ simultaneously, $O = \left(\frac{b}{4}, \frac{c}{4}\right)$. Since $O = \left(\frac{b}{4}, \frac{c}{4}\right)$ satisfies the equation $y - \frac{3c + \sqrt{3}b}{12} = \frac{b}{c} \left(x - \frac{3b + \sqrt{3}c}{12}\right)$ of the line $\overrightarrow{A^*A^{**}}$, then lines $\overrightarrow{A^*A^{**}}$, $\overrightarrow{B''B^{**}}$, and $\overrightarrow{C^{**}C^{**}}$ are concurrent at the point $O$. 

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Figure 3.3: \( \triangle ABC \) is a right triangle

2. The results follow since \( \overrightarrow{AB} \) and \( \overrightarrow{B**B} \) are horizontal lines and \( \overrightarrow{AC} \) and \( \overrightarrow{C**C} \) are vertical lines; and the slopes of \( \overrightarrow{BC} \) and \( \overrightarrow{A\hat{A}} \) are \(-\frac{c}{b}\) and \(\frac{c}{b}\), respectively.

3. The result follows from the statements (1) and (2) above.

\( \Leftrightarrow \) The result follows from conditions (3) and then (2)(figure 3.4).
Figure 3.4: $\triangle ABC$ is a right triangle
REFERENCES


