MONOTONIC SOLUTIONS OF SYSTEMS OF NONLINEAR DIFFERENTIAL EQUATIONS

by

Tyler J Myers

An Abstract
of a thesis submitted in partial fulfillment
of the requirements for the degree of
Master of Science
in the School of Computer Science and Mathematics
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Systems of nonseparable nonlinear differential equations can be hard to work with, but with certain additional conditions, we can acquire answers about continuability, existence, and boundedness of monotonic solutions. Specifically, we study systems of nonseparable first order nonlinear differential equations of the form $x'(t) = F(t, y(t)), y'(t) = G(t, x(t))$. First, we start by proving that all solutions of this system are eventually monotonic and can be sorted into one of four classes. Next, we discuss the continuability of all solutions using a further set of assumptions. Then we prove that all solutions are bounded provided that $|x'(t)| \leq p(t)f(y(t)), |y'(t)| \leq q(t)g(x(t))$ for some functions $p, f, q, \text{and } g$. Finally, we look at existence of solutions for two subclasses of such systems using the Schauder Fixed-Point Theorem.
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CHAPTER 1
INTRODUCTION AND MOTIVATION

We begin by introducing the systems of nonlinear differential equations that we will study and by recalling fundamental information about differential equations. We then list definitions that pertain to both differential equations and real analysis. Finally, we will state a few theorems that are important as we move through this paper.

1.1 Introduction

This thesis considers systems of nonseparable first order differential equations of the following form:

\[
\begin{align*}
    x'(t) &= F(t, y(t)) \\
    y'(t) &= G(t, x(t)).
\end{align*}
\]  

(1.1)

Throughout the thesis we assume that the functions \( F(t, r) \) and \( G(t, r) \) satisfy the conditions:

\[
\begin{align*}
    F(t, r) &\colon \mathbb{R} \to \mathbb{R} \text{ is continuous and } rF(t, r) > 0 \text{ for } r \neq 0; \\
    G(t, r) &\colon \mathbb{R} \to \mathbb{R} \text{ is continuous and } rG(t, r) > 0 \text{ for } r \neq 0; \\
    |F(t, y(t))| &\leq p(t)|f(y(t))| \text{ for some functions } p \text{ and } f; \\
    |G(t, x(t))| &\leq q(t)|g(x(t))| \text{ for some functions } q \text{ and } g; \\
    p(t), q(t) &\colon [a, \infty) \to \mathbb{R} \text{ continuous with } p(t) > 0, \; q(t) > 0; \\
    g(r) &\colon \mathbb{R} \to \mathbb{R} \text{ is continuous and } rg(r) > 0 \text{ for } r \neq 0; \\
    f(r) &\colon \mathbb{R} \to \mathbb{R} \text{ is continuous and } rf(r) > 0 \text{ for } r \neq 0 \\
    f(r) \text{ and } g(r) &\text{ are invertible}
\end{align*}
\]

Note that the assumptions \( |F(t, y(t))| \leq p(t)|f(y(t))| \) and \( |G(t, x(t))| \leq q(t)|g(x(t))| \) will need to be restricted further in Chapters 4 and 5. We will state the restriction in Chapter
CHAPTER 1. INTRODUCTION AND MOTIVATION

4 that will allow us to study the boundedness and existence of solutions.

Some kinds of systems of nonseparable nonlinear differential equations can be hard to work with, especially those in general form, like those from system (1.1). Separable equations are much easier to work with. Thus, we consider two additional conditions:

\[ |x'(t)| \leq p(t)f(y(t)), |y'(t)| \leq q(t)g(x(t)). \]

These conditions allow us to consider continuability, existence, and boundedness of monotonic solutions. Throughout, we heavily rely on work in [10].

The purpose of this thesis is to consider the continuability, classification, boundedness, and existence of solutions of system (1.1). As stated earlier, this chapter sets the stage for this thesis by introducing the system of first order differential equations and conditions for this system as well as definitions and theorems from differential equations and real analysis that pertain to the proofs in this thesis. Chapter 2 lists additional assumptions that apply to first order differential equations. In the second section of Chapter 2, we discuss four classifications of solutions of differential equations that our system can fall into. In Chapter 3 we consider continuability of solutions of this system. We investigate the boundedness of solutions in Chapter 4. Finally, in Chapter 5, we discuss the existence of solutions of this system for two subclasses of our classifications from Chapter 2.

1.2 Definitions Pertaining to Differential Equations

It is important in the beginning to list some definitions that will be used in our discussions. We state them without detailed explanation and leave them to the readers to check books of differential equations such as [13] if they desire to know more.

**Definition 1.2.1.** A pair \((x(t), y(t))\) is a solution of system (1.1) if both \(x(t)\) and \(y(t)\) are differentiable and satisfy system (1.1) on \([a, \alpha]\), where \([a, \alpha]\) is the maximal existence interval
solution of \((x(t), y(t))\).

**Remark 1.2.2.** The maximal existence interval \([a, \alpha]\) could be bounded, \(\alpha < \infty\), or unbounded, \(\alpha = \infty\).

**Definition 1.2.3.** A solution \((x(t), y(t))\) of system (1.1) is said to be eventually monotonic if there exists \(t_1 \geq a\) such that both \(x(t)\) and \(y(t)\) are monotonic on \([t_1, \alpha]\).

**Definition 1.2.4.** A solution \((x(t), y(t))\) of system (1.1) is said to be eventually identical to zero if there exists \(t_1 \geq a\) such that \((x(t), y(t)) \equiv (0, 0)\) on \([t_1, \alpha]\).

**Theorem 1.2.5** (Continuabilwity Theorem). Suppose system (1.1) has a unique local solution for each pair of initial conditions \((x(a), y(a)) = (x_0, y_0)\). Then there exists a unique solution defined on a maximal interval \((a, \alpha)\) which is called the maximal solution.

### 1.3 Definitions Pertaining to Real Analysis

This section lists definitions and theorems from real analysis that are referenced and used in this thesis. Similarly to the last section, we just state the definitions and theorems without detailed explanations. If more detail is desired, most of these definitions can be found in most textbooks such as [1].

**Definition 1.3.1.** A real vector space is a set \(X\) together with two operations

- **vector addition:** any two vectors \(x\) and \(y\) of \(X\) can be added to yield a third vector \(x + y\);

- **scalar multiplication:** any vector \(x\) of \(X\) can be multiplied by a real number \(a\) to get a new vector \(ax\),

that satisfy the following properties. Let \(x, y, z\) be arbitrary vectors in \(X\), and \(a, b\) be real numbers, respectively. Then
1. \( z + (x + y) = (z + x) + y; \)

2. \( x + y = y + x; \)

3. There exists an element \( 0 \in X \), called the zero vector, such that \( x + 0 = x \) for all \( x \in X \);

4. For each \( x \in X \), there exists an element \( y \in X \), called the additive inverse of \( x \), such that \( x + y = 0 \);

5. \( a(x + y) = ax + ay; \)

6. \( (a + b)x = ax + bx; \)

7. \( a(bx) = (ab)x. \)

**Definition 1.3.2.** Let \( S \) be a set. A function \( \| \cdot \| : S \to \mathbb{R} \) is called a norm on a vector space \( S \) if it has the following properties:

1. \( \| x \| \geq 0 \) for all \( x \in S; \)

2. \( \| x \| = 0 \) if and only if \( x = 0; \)

3. For any real number \( c \), \( \| cx \| = |c| \| x \| \) \( x \in S; \)

4. \( \| x + y \| \leq \| x \| + \| y \| \) for all \( x, y \in S. \)

**Definition 1.3.3.** A normed vector space, also called a normed linear space, is a real vector space \( X \) with a norm function \( \| \cdot \| \) defined on \( X. \)

**Definition 1.3.4.** Let \( X \) be a normed linear space and \( \{x_n\} \) a sequence in \( X \). If for any \( \epsilon > 0 \), there exists an integer \( N \) such that \( \| x_n - x_m \| \leq \epsilon \) whenever \( n, m \geq N \), we say that \( \{x_n\} \) is a Cauchy sequence in \( X. \)
Definition 1.3.5. A normed linear space $X$ is called a Banach space if it is complete. In other words, every Cauchy sequence in $X$ converges in $X$.

Definition 1.3.6. Let $D$ be a set of continuous functions defined on an interval $I$. We say that the set $D$ is uniformly bounded if there exists a real number $M > 0$ such that $|x(t)| \leq M$ for all $x \in D$ and all $t \in I$.

Definition 1.3.7. Let $D$ be a set of continuous functions defined on an interval $I$. We say that the set $D$ is equicontinuous on $I$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x(t_1) - x(t_2)| < \varepsilon$ for any $x \in D$ and $t_1, t_2 \in I$ such that $|t_1 - t_2| < \delta$.

Definition 1.3.8. Let $X$ be a Banach space and $D$ a subset of $X$. If every sequence in $D$ has a convergent subsequence in $D$, we say that $D$ is a compact set.

Definition 1.3.9. Let $X$ be a Banach space and $D$ a subset of $X$. If the closure of $D$ is compact, we say that $D$ is pre-compact.

Definition 1.3.10. Let $X$ be a Banach space and $D$ a subset of $X$. $D$ is said to be convex if for all $x$ and $y$ in $D$ and all $t \in [0, 1]$, the point $(1 - t)x + ty \in D$.

Definition 1.3.11. Let $I$ be an interval in $\mathbb{R}$ and $\{x_n\}$ be a sequence of real-valued functions defined on $I$. We say that $\{x_n\}$ uniformly converges to the limit $x : I \to \mathbb{R}$ if for every $\varepsilon > 0$, there exists a natural number $N$ such that $|x_n(t) - x(t)| < \varepsilon$ for all $t \in I$ and all $n \geq N$.

Theorem 1.3.12 (Lebesgue Dominated Convergence Theorem). Let $\{f_n(t)\}$ be a sequence of continuous functions defined on $[b, \infty)$ that satisfies the following conditions:

1. $\lim_{n \to \infty} f_n(t) = 0$ for all $t \in [b, \infty)$,
2. there exists a function $F(t)$ such that

$$|f_n(t)| \leq F(t), \quad \forall t \in [b, \infty),$$

and

$$\int_b^\infty F(t) dt < \infty.$$

Then

$$\lim_{n \to \infty} \int_b^\infty f_n(t) dt = 0.$$

**Theorem 1.3.13** (Schauder Fixed-Point Theorem). Let $X$ be a Banach space and $D$ be a closed convex subset of $X$. Assume that there exists a continuous mapping $F : D \to D$ such that $F(D)$ is pre-compact. Then $F$ has a fixed point in $D$.

**Theorem 1.3.14** (Arzelà–Ascoli Theorem). Let $[a, b]$ be a closed and bounded interval of $\mathbb{R}$ and $\{x_n\} \in C[a, b]$ for all $n \in \mathbb{N}$. If $\{x_n\}$ is uniformly bounded and equicontinuous on $[a, b]$ for all $n \in \mathbb{N}$, then there exists a subsequence of $\{x_n\}$ that converges uniformly on $[a, b]$.

**Theorem 1.3.15** (Diagonal Rule). Let $\{x_n\}$ be a sequence of continuous and bounded functions defined on $[a, \infty)$. If for each closed and bounded interval $I$ of $[a, \infty)$, $\{x_n\}$ has a subsequence that converges uniformly on $I$, then $\{x_n\}$ has a subsequence that converges uniformly on $[a, \infty)$.

We prove a standard proposition that will be used in Chapter 3.

**Proposition 1.3.16.** If $f(t)$ is continuous on $[a, \infty)$ and $\lim_{t \to \infty} f(t) = L$, then there exists $L_1 > 0$, such that $f(t) \leq L_1$ for all $t \in [a, \infty)$.

**Proof.** If $\lim_{t \to \infty} f(t) = L$, then for all $\epsilon > 0$ there exists $t^*$ such that $|f(t) - L| \leq \epsilon$. Whenever
$t \geq t^*$ with $\epsilon = \frac{|L|}{2}$, we have

$$f(t) < L - \epsilon < L + \frac{|L|}{2} = \frac{3|L|}{2}.$$ 

With

$$L_1 = \max\left\{ \frac{3|L|}{2}, \max_{\omega \leq t \leq \omega^*} |f(t)| \right\},$$

we have

$$f(t) \leq L_1.$$
2.1 Basic Assumptions

In this section we list all assumptions that will be used throughout the paper. Please note that these definitions primarily refer to the functions \( f(r) \) and \( g(r) \) defined in Chapter 1.

(H1) There exists \( M > 0 \) such that

\[
|f(uv)| \leq M|f(u)||f(v)| \quad \forall u, v \in \mathbb{R}.
\]

(H2) There exists \( N > 0 \) such that

\[
|g(uv)| \leq N|g(u)||g(v)| \quad \forall u, v \in \mathbb{R}.
\]

(H3) There exists \( m > 0 \) such that

\[
(u - v)(f(u) - f(v)) \geq 0 \quad \forall u, v : |u| \geq m, |v| \geq m.
\]

(H4) There exists \( n > 0 \) such that

\[
(u - v)(g(u) - g(v)) \geq 0 \quad \forall u, v : |u| \geq n, |v| \geq n.
\]

(H5) There exists \( r_0 > 0 \) such that

\[
\int_{r_0}^{\infty} \frac{dr}{f(g(r))} = \infty
\]
and
\[ \int_{-\infty}^{-r_0} \frac{dr}{f(g(r))} = -\infty. \]

(H6) There exists \( r_1 > 0 \) such that
\[ \int_{r_1}^{\infty} \frac{dr}{g(f(r))} = \infty \]

and
\[ \int_{-\infty}^{-r_1} \frac{dr}{g(f(r))} = -\infty. \]

Note that we only consider eventually nontrivial solutions \((x, y)\) of (1.1). In other words, both \(x\) and \(y\) are not eventually identically equal to zero.

2.2 Classification of Solutions

In this section, we discuss the classification of all solutions of system (1.1). There are four classes that the solutions can be placed into.

**Theorem 2.2.1.** If \((x, y)\) is a solution of system (1.1) with maximal existence interval \([a, \alpha)\), \(a < \alpha \leq \infty\), then \((x, y)\) is eventually monotonic and belongs to one of the following four classes:
\[ A = \left\{ (x, y) : \exists b \geq a \text{ such that } x(t) > 0, x'(t) > 0, y(t) > 0, y'(t) > 0 \right\} \text{ for all } t \in [b, \alpha) \], \\
\[ B = \left\{ (x, y) : \exists b \geq a \text{ such that } x(t) < 0, x'(t) < 0, y(t) < 0, y'(t) < 0 \right\} \text{ for all } t \in [b, \alpha) \], \\
\[ C = \left\{ (x, y) : x(t) > 0, x'(t) < 0, y(t) < 0, y'(t) > 0 \text{ for all } t \in [a, \infty) \right\} \], \\
\[ D = \left\{ (x, y) : x(t) < 0, x'(t) > 0, y(t) > 0, y'(t) < 0 \text{ for all } t \in [a, \infty) \right\} . \]

**Proof.** Let \( Q(t) = x(t)y(t) \). Then 

\[ Q'(t) = x'(t)y(t) + x(t)y'(t) \]

\[ = y(t)F(t, y(t)) + x(t)G(t, x(t)) \geq 0, \]

which implies that \( Q(t) \) is nondecreasing on \([a, \alpha)\). Hence, there are three cases for \( Q(t) \):

1. \( Q(t) < 0 \) for all \( t \in [a, \infty) \);
2. There exists \( b \geq a \) such that \( Q(t) > 0 \) for \( t \in [b, \alpha) \);
3. \( Q(t) \equiv 0 \) for all \( t \in [a, \alpha) \).

For case one, if \( Q(t) < 0 \) for all \( t \in [a, \infty) \), then \( x(t)y(t) < 0 \) for all \( t \in [a, \infty) \). Suppose that \( x(t) > 0 \), and \( y(t) < 0 \) for all \( t \in [a, \infty) \). Thus \( rG(t, r) > 0 \) implies \( x(t)G(t, x(t)) > 0 \) and thus \( y'(t) = G(t, x(t)) > 0 \). Also, \( rF(t, r) > 0 \) implies \( y(t)F(t, y(t)) > 0 \) and thus \( x'(t) = F(t, y(t)) < 0 \). Therefore, \((x, y) \in C\). Now, suppose that \( x(t) < 0, y(t) > 0 \) for all \( t \in [a, \infty) \). Then \( rG(t, r) > 0 \) implies \( x(t)G(t, x(t)) > 0 \) and thus \( y'(t) = G(t, x(t)) < 0 \). Also, \( rF(t, r) > 0 \) implies \( y(t)F(t, y(t)) > 0 \) and thus \( x'(t) = F(t, y(t)) > 0 \). Therefore, \((x, y) \in D\).
CHAPTER 2. CLASSIFICATION OF SOLUTIONS

For case two, if \( Q(t) > 0 \) for all \( t \in [b, \alpha) \), then \( x(t)y(t) > 0 \) for all \( t \in [b, \alpha) \). If \( x(t) > 0 \), \( y(t) > 0 \) for all \( t \in [b, \alpha) \), then \( y'(t) = G(t, x(t)) > 0 \) for all \( t \in [b, \alpha) \) because \( rG(t, r) > 0 \) which implies \( x(t)G(t, x(t)) > 0 \). We also have \( x'(t) = F(t, y(t)) > 0 \) for all \( t \in [b, \alpha) \) since \( rF(t, r) > 0 \) and thus \( y(t)F(t, y(t)) > 0 \). Therefore \( (x, y) \in A \). If \( x(t) < 0 \), \( y(t) < 0 \) for all \( t \in [b, \alpha) \), then \( y'(t) = G(t, x(t)) < 0 \) for all \( t \in [b, \alpha) \) because \( rG(t, r) > 0 \) which implies \( x(t)G(t, x(t)) > 0 \). Also, \( x'(t) = F(t, y(t)) < 0 \) for all \( t \in [b, \alpha) \) since \( rF(t, r) > 0 \) which implies \( y(t)F(t, y(t)) > 0 \). Then, \( (x, y) \in B \).

For case three, if \( Q(t) \equiv 0 \) for all \( t \in [a, \alpha) \), then \( (x(t), y(t)) \equiv (0, 0) \) eventually. In fact, suppose that there exists \( t_1 \in [a, \alpha) \) such that \( x(t_1) \neq 0 \). Since \( Q(t) \equiv 0 \), we know \( y(t_1) = 0 \). Since \( x(t) \) is continuous and \( x(t_1) \neq 0 \), there exists a neighborhood of \( t_1 \), say \( U_1 \), such that \( x(t) \neq 0 \) for all \( t \in U_1 \). In this case, we have

\[
y'(t) = G(t, x(t)) \neq 0
\]

since \( rg(r) > 0 \) and \( x(t) \neq 0 \) for all \( t \in U_1 \). However, \( y(t) \equiv 0 \) for all \( t \in U_1 \), which contradicts \( y'(t) \neq 0 \) for all \( t \in U_1 \). Therefore, \( x(t) \equiv 0 \) for all \( t \in [a, \alpha) \).

Now, assume there exists \( t_2 \in [a, \alpha) \) such that \( y(t_2) \neq 0 \). We then have \( x(t_2) = 0 \) because \( F(t) \equiv 0 \). Since \( y(t) \) is continuous, this implies that there exists a neighborhood of \( t_2 \), say \( U_2 \), such that \( y(t) \neq 0 \) for all \( t \in U_2 \). Thus since \( Q(t) \equiv 0 \), \( x(t) \equiv 0 \) for all \( t \in U_2 \). This contradicts the fact that

\[
x'(t) = F(t, y(t)) \neq 0
\]

for all \( t \in U_2 \), and therefore \( y(t) \equiv 0 \) for all \( t \in [a, \alpha) \).
Although the maximal existence interval for system (1.1) could be bounded, we want to determine if the maximal existence interval can be extended to $[a, \infty)$. This is also called the continuability of solutions. Using our assumptions from Chapter 2, we can prove the continuability of solutions of systems (1.1).

3.1 Continuability of Solutions

**Theorem 3.1.1.** Suppose that conditions (H1)-(H6) hold. Then all solutions $(x(t), y(t))$ of system (1.1) can be extended to $[a, \infty)$.

**Proof.** First, suppose $\alpha < \infty$, and that $[a, \alpha)$ is the maximum existence interval of $(x(t), y(t))$. We know from Theorem 2.2.1, that $(x, y)$ belongs to one of the four classes. Obviously, all solutions in classes $C$ and $D$ can be extended to $[a, \alpha)$. Without loss of generality we assume that $(x, y) \in A$. If $\lim_{t \to \alpha^-} x(t) = x_\alpha$ is finite, then $(x, y)$ can be extended at least to $[a, \alpha]$. By Continuability Theorem 1.2.5, $(x, y)$ can be extended further to a right neighborhood of $\alpha$ because $[a, \alpha]$ is obviously closed and bounded. This contradicts the assumption that $[a, \alpha)$ is the maximal existence interval. Then $\lim_{t \to \alpha^-} x(t) = \infty$ because $x(t)$ is strictly increasing in a left neighborhood of $\alpha$. Therefore, by (H4), there exist real numbers $n > 0$ and $c > a$ such that $x(c) = n$, and $x(t) \geq n$ for all $c \leq t < \alpha$. Integrating $y'(t) = G(t, x(t))$ from $c$ to $t$ gives

$$y(t) = y(c) + \int_c^t G(s, x(s))ds$$

$$\leq y(c) + \int_c^t q(s)g(x(s))ds.$$  

Also, $x(t) - x(c) \geq 0$ as $x(t)$ is increasing. By (H4), we have $g(x(t)) - g(x(c)) \geq 0$, which
then implies that $g(x(t)) \geq g(x(c))$. This means that

$$y(t) \leq y(c) + g(x(t)) \int_c^t q(s)ds$$

$$= g(x(t)) \left( \frac{y(c)}{g(x(t))} + \int_c^t q(s)ds \right).$$

Since $g(x(t)) \geq g(x(c))$, we know $\frac{1}{g(x(c))} \geq \frac{1}{g(x(t))}$, and then

$$y(t) \leq g(x(t)) \left( \frac{y(c)}{g(x(c))} + \int_c^t q(s)ds \right).$$

We must now consider two possibilities; when $\int_c^\infty q(s)ds$ is infinite, and when it is finite. If

$$\int_c^\infty q(s)ds = \infty,$$

then

$$\lim_{t \to \infty} \frac{y(c)}{g(x(c))} \int_c^t q(s)ds = 0.$$  

From Proposition 1.3.16, there exists a $L_1 > 0$ such that

$$\frac{y(c)}{g(x(c))} \leq L_1 \int_c^t q(s)ds.$$

Then

$$\frac{y(c)}{g(x(c))} \leq L_1 \int_c^t q(s)ds$$

and

$$\frac{y(c)}{g(x(c))} + \int_c^t q(s)ds \leq (L_1 + 1) \int_c^t q(s)ds.$$
If
\[ \int_c^{\infty} q(s) \, ds < \infty, \]
then
\[ \lim_{t \to \infty} \frac{y(c) - y(x(c))}{\int_c^{t} q(s) \, ds} = \frac{y(c) - y(x(c))}{\int_c^{\infty} q(s) \, ds}. \]
Again, by Proposition 1.3.16, we also can choose \( L_1 > 0 \) such that
\[ \frac{y(c)}{\int_c^{t} q(s) \, ds} \leq L_1, \]
which implies that
\[ \frac{y(c)}{g(x(c))} \leq L_1 \int_c^{t} q(s) \, ds. \]
Therefore
\[ \frac{y(c)}{g(x(c))} + \int_c^{t} q(s) \, ds \leq (L_1 + 1) \int_c^{t} q(s) \, ds. \]
Since \( q(t) > 0 \), we can choose \( k > 1 \), and \( t_1 \geq c \) such that for \( t \geq t_1 \)
\[ y(t) \leq kg(x(t)) \int_c^{t} q(s) \, ds. \]
Note that since
\[ x'(t) = F(t, y(t)) \leq p(t)f(y(t)), \]
this implies that
\[ \frac{x'(t)}{p(t)} = \frac{F(t, y(t))}{p(t)} \leq f(y(t)), \]
and then we have
\[ f^{-1}\left( \frac{x'(t)}{p(t)} \right) = f^{-1}\left( \frac{F(t, y(t))}{p(t)} \right) \leq y(t). \]
This inequality gives us
\[
f^{-1}\left(\frac{x'(t)}{p(t)}\right) \leq kg(x(t)) \int_c^t q(s)ds.
\]
Evaluating \( f \) on both sides of the inequality implies
\[
\frac{x'(t)}{p(t)} \leq f\left(kg(x(t)) \int_c^t q(s)ds\right).
\]
Using (H1) twice on the right side gives
\[
\frac{x'(t)}{p(t)} \leq M^2 f(k) f\left(g(x(t)) f\left(\int_c^t q(s)ds\right)\right).
\]
Then
\[
\frac{x'(t)}{f(g(x(t)))} \leq M^2 f(k) p(t) f\left(\int_c^t q(s)ds\right).
\]
Now, integrating from \( t_1 \) to \( t \), we have
\[
\int_{x(t_1)}^{x(t)} \frac{dr}{f(g(r))} = \int_{t_1}^t \frac{x'(s)ds}{f(g(x(s))} \leq M^2 f(k) \int_{t_1}^t p(s) f\left(\int_c^s q(\sigma)d\sigma\right)ds. \tag{3.1}
\]
Taking the limit as \( t \to \alpha^- \) yields
\[
\int_{x(t_1)}^{\infty} \frac{dr}{f(g(r))} \leq M^2 f(k) \int_{t_1}^{\alpha} p(s) f\left(\int_c^s q(\sigma)d\sigma\right)ds.
\]
By (H5), we have
\[
\int_{x(t_1)}^{\infty} \frac{dr}{f(g(r))} = \infty.
\]
This then implies that
\[
\int_{t_1}^{\alpha} p(s) f\left(\int_c^s q(\sigma)d\sigma\right)ds = \infty,
\]
but this is impossible because \( \alpha < \infty \). Therefore \( x(t) \) can be extended to infinity.
Similarly, if \( \lim_{t \to a^-} y(t) = y_{a} \) is finite, then \((x, y)\) can be extended at least to \([a, \alpha]\). By Continuability Theorem 1.2.5, \((x, y)\) can be extended further to a right neighborhood of \(\alpha\) because \([a, \alpha]\) is closed and bounded. This contradicts the assumption that \([a, \alpha]\) is the maximal existence interval. Then \( \lim_{t \to a^-} y(t) = \infty \) because \(y(t)\) is strictly increasing in a left neighborhood of \(\alpha\). Therefore there exist real numbers \(m > 0\) from (H3), and \(c > a\) such that \(y(c) = m\), and \(y(t) \geq m\) for all \(c \leq t < \alpha\). Integrating \(x'(t) = F(s, y(s))\) from \(c\) to \(t\) gives

\[
x(t) = x(c) + \int_c^t F(s, y(s)) \, ds \\
\leq x(c) + \int_c^t p(s) f(y(s)) \, ds.
\]

Since \(y(c) = m\), and \(y(t) \geq m\) for all \(c \leq t < \alpha\), we can say that \(y(t) - y(c) \geq 0\) as \(y(t)\) is increasing. By (H3), we know that \(f(y(t)) - f(y(c)) \geq 0\) which then implies that \(f(y(t)) \geq f(y(c))\). This means that

\[
x(t) \leq x(c) + f(y(t)) \int_c^t p(s) \, ds \\
= f(y(t)) \left( \frac{x(c)}{f(y(t))} + \int_c^t p(s) \, ds \right).
\]

Since \(f(y(t)) \geq f(y(c))\), we have that \(\frac{1}{f(y(c))} \geq \frac{1}{f(y(t))}\), and then

\[
x(t) \leq f(y(t)) \left( \frac{x(c)}{f(y(c))} + \int_c^t p(s) \, ds \right).
\]

If

\[
\int_c^\infty p(s) \, ds = \infty,
\]
CHAPTER 3. CONTINUABILITY OF SOLUTIONS

then

\[
\lim_{t \to \infty} \frac{x(c)}{\int_c^t f(y(c)) p(s) ds} = 0.
\]

We can choose \( L_1 > 0 \) using Proposition 1.3.16 such that

\[
\frac{x(c)}{\int_c^t f(y(c)) p(s) ds} \leq L_1.
\]

Then

\[
\frac{x(c)}{f(y(c))} \leq L_1 \int_c^t p(s) ds
\]

and

\[
\frac{x(c)}{f(y(c))} + \int_c^t p(s) ds \leq (L_1 + 1) \int_c^t p(s) ds.
\]

If

\[
\int_c^\infty q(s) ds < \infty,
\]

then

\[
\lim_{t \to \infty} \frac{x(c)}{\int_c^t f(y(c)) p(s) ds}.
\]

As we have done previously, by using Proposition 1.3.16, we can find \( L_1 \) such that

\[
\frac{x(c)}{f(y(c))} \leq L_1,
\]

which implies that

\[
\frac{x(c)}{f(y(c))} \leq L_1 \int_c^t p(s) ds.
\]

Thus,

\[
\frac{x(c)}{f(y(c))} + \int_c^t p(s) ds \leq (L_1 + 1) \int_c^t p(s) ds.
\]
As \( p(t) > 0 \), there exists \( k > 1 \), and \( t_1 \geq c \) such that for \( t \geq t_1 \)

\[
x(t) \leq kf(y(t)) \int_c^t p(s)ds.
\]

Since we know

\[
y'(t) = G(t, x(t)) \leq q(t)g(x(t)),
\]

then

\[
\frac{y'(t)}{q(t)} = \frac{G(t, x(t))}{q(t)} \leq g(x(t)),
\]

so we can say

\[
g^{-1}\left(\frac{y'(t)}{q(t)}\right) = g^{-1}\left(\frac{G(t, x(t))}{q(t)}\right) \leq x(t).
\]

We can then imply that

\[
g^{-1}\left(\frac{y'(t)}{q(t)}\right) \leq kf(y(t)) \int_c^t p(s)ds.
\]

Next, evaluating \( g \) on both sides of the inequality yields

\[
\frac{y'(t)}{q(t)} \leq g \left( kf(y(t)) \int_c^t p(s)ds \right).
\]

By using (H2) twice, we have

\[
\frac{y'(t)}{q(t)} \leq N^2 g(k)g( f(y(t)) ) g \left( \int_c^t p(s)ds \right).
\]

Thus

\[
\frac{y'(t)}{g(f(y(t)))} \leq N^2 g(k)q(t)g \left( \int_c^t p(s)ds \right).
\]
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If we integrate from $t_1$ to $t$, we get

$$\int_{y(t_1)}^{y(t)} \frac{dr}{g(f(r))} = \int_{t_1}^{t} \frac{y'(s)ds}{g(f(y(s)))} \leq N^2 g(k) \int_{t_1}^{t} q(s) g\left( \int_{c}^{s} p(\sigma)d\sigma \right)ds. \quad (3.2)$$

Taking the limit $t \to \alpha^-$, we have

$$\int_{y(t_1)}^{\infty} \frac{dr}{g(f(r))} \leq N^2 g(k) \int_{t_1}^{\alpha} q(s) g\left( \int_{c}^{s} p(\sigma)d\sigma \right)ds.$$

By (H6), we have

$$\int_{y(t_1)}^{\infty} \frac{dr}{g(f(r))} = \infty.$$

This then implies that

$$\int_{t_1}^{\alpha} q(s) g\left( \int_{c}^{s} p(\sigma)d\sigma \right)ds = \infty,$$

but this is impossible because $\alpha < \infty$. Therefore, $y(t)$ can be extended to infinity.

Altogether, we have proved that the maximal existence interval of the solution $(x(t), y(t))$ is $[a, \infty)$.
CHAPTER 4

BOUNDEDNESS OF SOLUTIONS

We will now prove the boundedness of solutions of system (1.1). We start by proving the boundedness of \( x(t) \) and \( y(t) \) separately. Finally, we combine these proofs and theorems so that we can prove the boundedness of solutions \((x, y)\). For the sake of the following theorems, we need to impose additional assumptions for \( F(t, y(t)) \) and \( G(t, x(t)) \). In particular we assume:

\[
p_1(t)|f_1(r)| \leq |F(t, r)| \leq p_2(t)|f_2(r)| \text{ and } \quad q_1(t)|g_1(r)| \leq |G(t, r)| \leq q_2(t)|g_2(r)|.
\]

as well as

\[
\begin{align*}
p_i(t), q_i(t) &: [a, \infty) \to \mathbb{R} \text{ are continuous;} \\
p_i(t) > 0, q_i(t) > 0 & \text{ for } i = 1, 2; \\
g_i(r) &: \mathbb{R} \to \mathbb{R} \text{ is continuous and } rg_i(r) > 0 \text{ for } r \neq 0 \text{ and } i = 1, 2; \\
f_i(r) &: \mathbb{R} \to \mathbb{R} \text{ is continuous and } rf_i(r) > 0 \text{ for } r \neq 0 \text{ and } i = 1, 2; \text{ and} \\
f_i(r) \text{ and } g_i(r) & \text{ are invertible for } i = 1, 2
\end{align*}
\]

We make use the following two improper integrals to characterize the properties of solutions of system (1.1)

\[
J_{1i} := \int_a^\infty p_i(t)f_i\left(\int_a^t q_i(s)ds\right)dt \text{ for } i = 1, 2,
\]

\[
J_{2i} := \int_a^\infty q_i(t)g_i\left(\int_a^t p_i(s)ds\right)dt \text{ for } i = 1, 2.
\]
4.1  Boundedness of Solutions

**Theorem 4.1.1.** Suppose that conditions (H1), (H4), and (H5) hold. Then the $x(t)$ components of all solutions $(x, y)$ of (1.1) are bounded if and only if $J_{1i} < \infty$ for $i = 1, 2$.

**Proof.** We know that solutions from classes $C$ and $D$ are always bounded. Thus we need only to consider solutions from class $A$ and class $B$. We will only prove boundedness of solutions from class $A$ as the proof will be very similar for solutions of class $B$. Let $(x, y)$ be a class $A$ solution to system (1.1).

Since $x(t)$ is bounded and $x(t) > 0$, $y(t) > 0$, $x'(t) > 0$, $y'(t) > 0$ for all $t \geq b \geq a$, we have $\lim_{t \to \infty} x(t) = L \in (0, \infty)$. Then

$$M_1 = \min_{x(b) \leq r \leq L} g_1(r) > 0.$$  

Let

$$x(\infty) := \lim_{t \to \infty} x(t) < \infty.$$ 

Integrating $y'(t) = G(t, x(t))$ from $b$ to $t$, we have

$$y(t) = y(b) + \int_b^t G(s, x(s))ds \geq y(b) + \int_b^t q_1(s)g_1(x(s))ds$$
$$\geq M_1 \int_b^t q_1(s)ds.$$ 

From system (1.1), we have

$$f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right) \geq y(t) \geq M_1 \int_b^t q_1(s)ds,$$

and so

$$\frac{1}{M_1}f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right) \geq \int_b^t q_1(s)ds.$$
CHAPTER 4. BOUNDEDNESS OF SOLUTIONS

Applying (H1) implies that

\[ Mf_1\left(\frac{1}{M_1}\right)x'(t) \geq p_1(t)f_1\left(\int_b^t q_1(s)ds\right). \]

Integrating this inequality from \(b\) to infinity yields

\[ \int_b^\infty p_1(s)f_1\left(\int_b^s q_1(\sigma)d\sigma\right)ds \leq Mf_1\left(\frac{1}{M_1}\right)(x(\infty) - x(b)) < \infty, \]

and hence,

\[ J_{11} = \int_a^\infty p_1(s)f_1\left(\int_a^s q_1(\sigma)d\sigma\right) < \infty. \]

Since (H1), (H4), and (H5) hold on the interval \([a, \infty)\), using an argument similar to the proof of Theorem 3.1.1, we see that

\[ \int_{x(t)}^{x(t_1)} \frac{dr}{f_2(g_2(r))} \leq M^2f_2(k) \int_{t_1}^{t} p_2(s)f_2\left(\int_c^s q_2(\sigma)d\sigma\right)ds \leq M^2f_2(k) \int_{t_1}^{\infty} p_2(s)f_2\left(\int_c^s q_2(\sigma)d\sigma\right)ds. \]

If \(x(t)\) is unbounded, then \(\lim_{t \to \infty} x(t) = \infty\) and

\[ \int_{x(t_1)}^{\infty} \frac{dr}{f_2(g_2(r))} \leq M^2f_2(k) \int_{t_1}^{\infty} p_2(s)f_2\left(\int_c^s q_2(\sigma)d\sigma\right)ds < \infty. \]

This is a contradiction to (H5). Therefore the \(x(t)\) component is bounded.

The next theorem provides the necessary and sufficient condition for the boundedness of all \(y(t)\) components.

**Theorem 4.1.2.** Suppose that conditions (H2), (H3), and (H6) hold. Then the \(y(t)\) components of all solutions \((x,y)\) of (1.1) are bounded if and only if \(J_{2i} < \infty\) for \(i = 1, 2\).
**Proof.** Similar to Theorem 4.1.1, we consider only class $A$ solutions.

Since $y(t)$ is bounded and $x(t) > 0$, $y(t) > 0$, $x'(t) > 0$, and $y'(t) > 0$ for all $t \geq b \geq a$, then $\lim_{t \to \infty} y(t) = L_2 \in (0, \infty)$. Define

$$M_2 = \min_{y(b) \leq r \leq L_2} f_1(r) > 0.$$ 

Integrating $x'(t) = F(t, y(t))$, we have

$$x(t) = x(b) + \int_b^t F(s, y(s))ds \geq x(b) + \int_b^t p_1(s)f_1(y(s))ds \geq M_2 \int_b^t p_1(s)ds.$$ 

Note that since

$$x(t) \leq g_1^{-1}\left(\frac{y'(t)}{q_1(t)}\right),$$

we have

$$g_1^{-1}\left(\frac{y'(t)}{q_1(t)}\right) \geq M_2 \int_b^t p_1(s)ds,$$

and so

$$\frac{1}{M_2} g_1^{-1}\left(\frac{y'(t)}{q_1(t)}\right) \geq \int_b^t p_1(s)ds.$$ 

By (H2), we obtain

$$Ng_1\left(\frac{1}{M_2}\right)y'(t) \geq q_1(t)g\left(\int_b^t p_1(s)ds\right).$$

Integrating from $b$ to infinity yields

$$\int_b^\infty q_1(s)g_1\left(\int_b^s p_1(\sigma)d\sigma\right)ds \leq Ng_1\left(\frac{1}{M_2}\right)(y(\infty) - y(b)) < \infty.$$ 

Therefore,

$$J_{21} = \int_a^\infty q_1(s)g_1\left(\int_a^s p_1(\sigma)d\sigma\right)ds < \infty.$$
Since (H2), (H3), and (H6) hold on the interval \([a, \infty)\), following the same logic as when proving the boundedness of the \(x(t)\) component, we have

\[
\int_{y(t_1)}^{y(t)} \frac{dr}{g_2(f_2(r))} \leq N^2 g_2(k^*) \int_{t_1}^{t} q_2(s)g_2\left(\int_{c^*}^{s} p_2(\sigma)d\sigma\right)ds \\
\leq N^2 g_2(k^*) \int_{t_1}^{\infty} q_2(s)g_2\left(\int_{c^*}^{s} p_2(\sigma)d\sigma\right)ds.
\]

If \(y(t)\) is unbounded, then \(\lim_{t \to \infty} y(t) = \infty\). Taking the limit as \(t \to \infty\), we have

\[
\int_{y(t_1)}^{\infty} \frac{dr}{g_2(f(r))} \leq N^2 g_2(k^*) \int_{t_1}^{\infty} q_2(s)g_2\left(\int_{c^*}^{s} p_2(\sigma)d\sigma\right)ds < \infty.
\]

This is a contradiction to (H6). Therefore the \(y(t)\) component is bounded. \(\square\)

**Theorem 4.1.3.** Suppose that conditions (H1)-(H6) hold. Then all solutions of (1.1) are bounded if and only if \(J_{1i} < \infty\) and \(J_{2i} < \infty\) for \(i = 1, 2\).

**Proof.** The proof follows from Theorem 4.1.1 and Theorem 4.1.2. \(\square\)
Finally, we will prove the existence of certain subclasses of class \( A \) and \( B \). To this end we define two subclasses

\[
A_b = \left\{ (x, y) \in A : \lim_{t \to \infty} x(t) = \ell < \infty, \lim_{t \to \infty} y(t) = \ell_1 < \infty \right\}
\]

and

\[
B_b = \left\{ (x, y) \in B : \lim_{t \to \infty} x(t) = \ell > -\infty, \lim_{t \to \infty} y(t) = \ell_1 > -\infty \right\}.
\]

First, we will establish the non-emptiness of subclass \( A_b \) and subclass \( B_b \) in the following two theorems.

5.1 Existence of Positive Solutions

The following results, Lemma 5.1.1, Proposition 5.1.2, and Proposition 5.1.3 from [10] give us the tools necessary to prove the existence of positive solutions. Lemma 5.1.1 will be used to prove the non-emptiness of subclass \( A_b \) and subclass \( B_b \).

**Lemma 5.1.1.** Suppose \((H1)\) and \((H4)\) hold. If

\[
\int_b^\infty p(s) f\left( \int_b^s q(\sigma) d\sigma \right) ds < \infty,
\]

then for any positive number \( S \), we can choose a number \( c \geq b \) such that

\[
\int_c^\infty p(s) f\left( S + \int_c^s q(\sigma) d\sigma \right) ds < \infty.
\]
Consider $CB[d, \infty)$ as the set of all bounded and continuous functions defined on $[d, \infty)$ with the norm

$$
\|x\| = \sup_{d \leq t < \infty} |x(t)|. \tag{5.1}
$$

**Proposition 5.1.2.** $CB[d, \infty)$ is a Banach space

Define a nonempty subset of $CB[d, \infty)$ as

$$
X = \{x \in CB[d, \infty) : 1 \leq x(t) \leq 2, \ t \geq d \}.
$$

Clearly, $X$ is bounded.

**Proposition 5.1.3.** $X$ is closed and convex.

Now we prove that system (1.1) has a solution in the subclass $A_b$.

**Theorem 5.1.4.** Suppose that (H1) and (H4) hold. Then system (1.1) has a solution in subclass $A_b$ if and only if $J_{1i} < \infty$ for $i = 1, 2$.

**Proof.** For necessity, let $(x, y)$ be a subclass $A_b$ solution. Then there exists $b \geq a$ such that $x(t) > 0$, $x'(t) > 0$, $y(t) > 0$, and $y'(t) > 0$ for all $t \geq b$. Note that

$$
x(\infty) := \lim_{t \to \infty} x(t) < \infty.
$$

Then

$$
m_1 := \min_{x(b) \leq r \leq x(\infty)} g_1(r) > 0
$$

by the Extreme Value Theorem and the assumption that $rg(r) > 0$. 

Integrating both sides of the equation \( y'(t) = G(t, x(t)) \) from \( b \) to \( t \), we have

\[
y(t) = y(b) + \int_b^t G(s, x(s)) \, ds \\
\geq y(b) + \int_b^t q_1(s) g_1(x(s)) \, ds \\
\geq y(b) + m_1 \int_b^t q_1(s) \, ds \\
\geq m_1 \int_b^t q_1(s) \, ds.
\]

From this inequality and since

\[
y(t) \leq f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right),
\]

we obtain

\[
\int_b^t q_1(s) \, ds \leq \frac{1}{m_1} f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right).
\]

Using (H1) and applying \( f \) to both sides yields

\[
p_1(t) f_1\left(\int_b^t q_1(s) \, ds\right) \leq M f_1\left(\frac{1}{m_1}\right) x'(t).
\]

Recall that \( x(t) \) is continuous and bounded on \([b, \infty)\). By integrating both sides from \( b \) to infinity we have

\[
\int_b^\infty p_1(t) f_1\left(\int_b^t q_1(s) \, ds\right) \, dt \leq M f_1\left(\frac{1}{m_1}\right) (x(\infty) - x(b)) < \infty.
\]

Then

\[
J_{11} = \int_a^\infty p_1(t) f_1\left(\int_a^t q_1(s) \, ds\right) \, dt < \infty.
\]

For sufficiency, let

\[
M_2 = \max_{1 \leq r \leq 2} g_2(r) > 0.
\]
Since

\[ J_{12} = \int_a^\infty p_2(t) f_2 \left( \int_a^t q_2(s) ds \right) dt < \infty, \]

by Lemma 5.1.1, we can choose \( d \geq b \) such that

\[ \int_d^\infty p_2(s)f_2 \left( \frac{1}{M_2} + \int_s^t q_2(\sigma) d\sigma \right) ds \leq \frac{1}{M f_2(M_2)}. \]

From Proposition 5.1.2 and Proposition 5.1.3, we know that \( CB[d, \infty) \) is a Banach space and that \( X \) is closed and convex.

Define a mapping \( Z_1 : X \to CB[d, \infty) \) by

\[ (Z_1x)(t) = 1 + \int_d^t F \left( s, 1 + \int_d^s G(\sigma, x(\sigma)) d\sigma \right) ds. \]

In order to apply Theorem 1.3.13 to show that \( Z_1 \) has a fixed point in \( X \), we need to show that

1. \( Z_1 \) maps \( X \) into \( X \);
2. \( Z_1 \) is continuous in \( X \); and
3. \( Z_1(X) \) is pre-compact in \( CB[d, \infty) \).

The proof of (1). Let \( x \in X \). Then \( 1 \leq x(t) \leq 2 \) for all \( t \in [d, \infty) \). Obviously, \( (Z_1x)(t) \geq 1 \) since

\[ \int_d^t F \left( s, 1 + \int_d^s G(\sigma, x(\sigma)) d\sigma \right) ds > 0. \]

Note that

\[ (Z_1x)(t) = 1 + \int_d^t F \left( s, 1 + \int_d^s G(\sigma, x(\sigma)) d\sigma \right) ds \leq 1 + \int_d^t p_2(s)f_2 \left( 1 + \int_d^s G(\sigma, x(\sigma)) d\sigma \right) ds. \]
The next-to-last inequality is due to the fact that
\[
\int_d^\infty p_2(s)f_2\left(\frac{1}{M_2} + \int_d^s q_2(\sigma)d\sigma\right)ds \leq \frac{1}{M_2f_2(M_2)}.
\]

It follows that \(Z_1x \in X\). Then \(Z_1\) maps \(X\) into \(X\).

The proof of (2). We show that if \(\{x_n\}\), there exists \(x^* \in X\) such that \(\|x_n - x^*\| \to 0\) as \(n \to \infty\), then \(\|Z_1x_n - Z_1x^*\| \to 0\) as \(n \to \infty\).

Indeed, for any \(s \in [d, \infty)\), since \(x_n(s) \to x^*(s)\) as \(n \to \infty\), we have that
\[
\left|F\left(\sigma, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(\sigma, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)\right| \to 0
\]
since \(F\) and \(G\) are continuous.

Also, for any \(s \in [d, \infty)\), we have
\[
\left|F\left(s, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)\right| \\
\leq \left|F\left(s, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right)\right| + \left|F\left(s, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)\right|
\]
\[ \leq \left| p_2(s)f_2\left(1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right)\right| + \left| p_2(s)f_2\left(1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)\right| \quad (5.3) \]
\[ \leq \left| p_2(s)f_2\left(1 + \int_d^s q_2(\sigma)g_2(x_n(\sigma))d\sigma\right)\right| + \left| p_2(s)f_2\left(1 + \int_d^s q_2(\sigma)g_2(x^*(\sigma))d\sigma\right)\right| \]
\[ \leq 2Mf_2(M_2)p_2(s)f_2\left(\frac{1}{M_2} + \int_d^s q_2(\sigma)d\sigma\right) := K(s). \]

and
\[ \int_d^\infty K(s)ds = \int_d^\infty 2Mf_2(M_2)p_2(s)f_2\left(\frac{1}{M_2} + \int_d^s q_2(\sigma)d\sigma\right)ds < \infty. \quad (5.4) \]

It follows from (5.3), (5.4), and Theorem 1.3.12 that

\[ \|Z_1x_n - Z_1x^*\| \]
\[ = \sup_{b \leq t < \infty} |(Z_1x_n)(t) - (Z_1x^*)(t)| \]
\[ = \sup_{b \leq t < \infty} \left| \int_d^t F\left(s, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right)ds - \int_d^t F\left(s, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)ds \right| \]
\[ \leq \sup_{b \leq t < \infty} \int_d^t \left| F\left(s, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right| ds \]
\[ \leq \int_d^\infty \left| F\left(s, 1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, 1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right| ds \to 0. \]

as \( n \to \infty \). Therefore \( Z_1 \) is continuous in \( X \).

The proof of (3). In order to prove that \( Z_1(X) \) is pre-compact in \( CB[d, \infty) \), we need to show that for any sequence \( \{x_n\} \in X \), \( \{Z_1x_n\} \) has a convergent subsequence in \( CB[d, \infty) \).

If we can prove that \( \{Z_1x_n\} \) has a convergent subsequence in \( C[b_1, b_2] \) for any closed and bounded interval \([b_1, b_2]\) of \([d, \infty)\), then \( \{Z_1x_n\} \) has a convergent subsequence in \( CB[d, \infty) \) by Theorem 1.3.15. From Theorem 1.3.14, to prove that \( \{Z_1x_n\} \) has a convergent subsequence in \( C[b_1, b_2] \) we need to show that \( \{Z_1x_n\} \) is uniformly bounded and equicontinuous in \( C[b_1, b_2] \).

Since \( \{x_n\} \in X \) and \( Z_1(X) \subset X \), we have \( 1 \leq (Z_1x_n)(t) \leq 2 \) for all \( t \in [d, \infty) \) and \( n \in \mathbb{N} \). Thus \( \{Z_1x_n\} \) is uniformly bounded.

To prove the equicontinuity on \([b_1, b_2]\), we need to show that for any \( \epsilon > 0 \), there exists
$\delta > 0$ such that
\[
| (Z_1 x_n)(t_1) - (Z_1 x_n)(t_2) | < \epsilon
\]
for any $n \in \mathbb{N}$ and $t_1, t_2 \in [b_1, b_2]$ such that $|t_1 - t_2| < \delta$.

Notice that
\[
(Z_1 x_n)'(t) = F\left( t, 1 + \int_{d}^{t} G(s, x_n(s)) ds \right)
\leq p_2(t) f_2 \left( 1 + \int_{d}^{t} q_2(s) g_2(x_n(s)) ds \right)
\leq M f_2(M_2) p_2(t) f_2 \left( \frac{1}{M_2} + \int_{d}^{t} q_2(s) ds \right).
\]
By the Mean Value Theorem, we know there exists $\xi \in [t_1, t_2]$ such that
\[
(Z_1 x_n)(t_1) - (Z_1 x_n)(t_2) = |(Z_1 x_n)'(\xi)(t_1 - t_2)|
\leq M f_2(M_2) \max_{b_1 \leq \xi \leq b_2} \left( p_2(t) f_2 \left( \frac{1}{M_2} + \int_{d}^{t} q_2(s) ds \right) \right) |t_1 - t_2|.
\]
Therefore, to make
\[
| (Z_1 x_n)(t_1) - (Z_1 x_n)(t_2) | < \epsilon,
\]
we can choose
\[
\delta = \frac{\epsilon}{M f_2(M_2) \max_{b_1 \leq \xi \leq b_2} \left( p_2(t) f_2 \left( \frac{1}{M_2} + \int_{d}^{t} q_2(s) ds \right) \right)}.
\]
Therefore, $\{Z_1 x_n\}$ is equicontinuous on $[b_1, b_2]$.

Now all conditions of Theorem 1.3.13 are satisfied. Hence, $Z_1$ has a fixed point $\bar{x}$ in $X$, that is $Z_1 \bar{x} = \bar{x}$.

Define
\[
\bar{y}(t) = 1 + \int_{d}^{t} G(s, \bar{x}(s)) ds.
\]
In the following we prove that \((\bar{x}(t), \bar{y}(t))\) is a subclass \(A_b\) solution of system (1.1).

Consider \(\bar{x}(t)\) such that

\[
\bar{x}(t) = 1 + \int_d^t F\left(s, 1 + \int_d^s G(s, \bar{x}(\sigma))d\sigma\right)ds.
\]

Differentiating both sides with respect to \(t\), we have

\[
\bar{x}'(t) = F\left(t, 1 + \int_d^t G(s, \bar{x}(s))ds\right)
= F(t, \bar{y}(t)).
\]

Also,

\[
\bar{y}'(t) = G(t, \bar{x}(t)).
\]

Therefore, \((\bar{x}, \bar{y})\) is a \(A_b\) solution of (1.1). \(\square\)

5.2 Existence of Negative Solutions

In this section we consider the existence of subclass \(B_b\) solutions of system (1.1).

Now, define a nonempty subset of \(CB[d, \infty)\) as

\[
X_2 = \{x \in CB[d, \infty) : -2 \leq x(t) \leq -1, \ t \geq d\}.
\]

Obviously, \(X_2\) is bounded.

From [10] we have the following Proposition.

**Proposition 5.2.1.** \(X_2\) is closed and convex.

Now we will use the results from Lemma 5.1.1, Proposition 5.1.2, and Proposition 5.1.1 to show that system (1.1) has a solution in the subclass \(A_b\).
Theorem 5.2.2. Suppose that (H1) and (H4) hold. Then system (1.1) has a solution in subclass $B_b$ if and only if $J_{1i} < \infty$ for $i = 1, 2$.

Proof. Assume that $(x, y)$ is subclass $B_b$ solution of system (1.1). Then there exists a $b \geq a$ such that $x(t) < 0$, $y(t) < 0$, $x'(t) < 0$, and $y'(t) < 0$ for $t \geq b$.

Define

$$M_1 = \max_{x(\infty) \leq r \leq x(b)} g_1(r).$$

Then $M_1 < 0$ by the Extreme Value Theorem and the assumption $rg(r) > 0$.

Integrating both sides of $y'(t) = G(t, x(t))$ from $b$ to $t$, we have

$$y(t) = y(b) + \int_b^t G(s, x(s))ds$$

$$\leq y(b) + \int_b^t q_1(s)g_1(x(s))ds \leq M_1 \int_b^t q_1(s)ds.$$ 

Note that

$$y(t) \geq f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right),$$

and so

$$\frac{1}{M_1} f_1^{-1}\left(\frac{x'(t)}{p_1(t)}\right) \geq \int_b^t q_1(s)ds.$$ 

It follows from (H1) that

$$M f_1\left(\frac{1}{M_1}\right) x'(t) \geq f_1\left(\int_b^t q_1(s)ds\right).$$

Multiplying through by $p_1(t)$ produces

$$p_1(t) f_1\left(\int_b^t q_1(s)ds\right) \leq M f_1\left(\frac{1}{M_1}\right) x'(t).$$
Integrating both sides from \( b \) to infinity implies

\[
\int_b^\infty p_1(s)f_1\left(\int_b^s q_1(\sigma)d\sigma\right)ds \leq Mf_1\left(\frac{1}{M_1}\right)(x(\infty) - x(b)) < \infty.
\]

Therefore,

\[
J_{11} = \int_a^\infty p_1(s)f_1\left(\int_a^s q_1(\sigma)d\sigma\right)ds < \infty.
\]

For sufficiency, consider

\[
M_3 = \min_{-2 \leq r \leq -1} g_2(r) < 0.
\]

From Lemma 5.1.1 we can choose \( d > b \) such that

\[
\int_d^\infty p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right)ds \leq -\frac{1}{Mf_2(M_3)}.
\]

By Proposition 5.1.2 and Proposition 5.2.1, \( CB[d, \infty) \) is a Banach space and \( X_2 \) is closed and convex.

Define a mapping \( Z_2 : X_2 \to CB[d, \infty) \) by

\[
(F_2x)(t) = -1 + \int_d^t F\left(s, -1 + \int_d^s G(\sigma, x(\sigma))d\sigma\right)ds.
\]

In order to apply Theorem 1.3.13 to show that \( Z_2 \) has a fixed point in \( X_2 \), we need to show that

1. \( Z_2 \) maps \( X_2 \) into \( X_2 \);
2. \( Z_2 \) is continuous in \( X_2 \); and
3. \( Z_2(X_2) \) is pre-compact in \( CB[d, \infty) \).
The proof of (1). Let $x \in X_2$. Then $-2 \leq x(t) \leq -1$, and $(Z_2x)(t) \leq -1$ because
\[
\int_d^t F\left(s, -1 + \int_d^s G(\sigma, x(\sigma))d\sigma\right)ds \leq 0.
\]
Also,
\[
(\mathcal{Z}_2x)(t) = -1 + \int_d^t F\left(s, -1 + \int_d^s G(\sigma, x(\sigma))d\sigma\right)ds
\geq -1 + \int_d^t Mf_2(M_3)p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right)ds
= -1 + Mf_2(M_3)\int_d^t p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right)ds
\geq -1 + Mf_2(M_3)\int_d^\infty \frac{1}{Mf_2(M_3)}
\geq -2.
\]
The next-to-last inequality follows from
\[
\int_d^\infty -p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right)ds \geq \frac{1}{Mf_2(M_3)}.
\]
Hence, $\mathcal{Z}_2(X_2) \subset X_2$, and therefore $\mathcal{Z}_2 : X_2 \to X_2$.

The proof of (2). We now show that if $\{x_n\} \in X_2$, there exists $x^* \in X_2$ such that $\|x_n - x^*\| \to 0$ as $n \to \infty$, then $\|\mathcal{Z}_2x_n - \mathcal{Z}_2x^*\| \to 0$ as $n \to \infty$, which implies the continuity of $\mathcal{Z}_2$ in $X_2$.

Note that for all $s \in [d, \infty)$, $x_n(s) \to x^*(s)$ as $n \to \infty$, so
\[
\left|F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)\right| \to 0 \quad (5.5)
\]
In addition, we have

\[
\left| F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right| \\
\leq \left| F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) \right| + \left| F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right| \\
\leq \left| p_2(s)f_2\left(-1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) \right| + \left| p_2(s)f_2\left(-1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right| \\
\leq \left| p_2(s)f_2\left(-1 + \int_d^s q_2(\sigma)g_2(x_n(\sigma))d\sigma\right) \right| + \left| p_2(s)f_2\left(-1 + \int_d^s q_2(\sigma)g_2(x^*(\sigma))d\sigma\right) \right| \\
\leq -2Mf_2(M_3)p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right) := K(s)
\]

and

\[
\int_d^\infty K(s)ds = \int_d^\infty -2Mf_2(M_3)p_2(s)f_2\left(-\frac{1}{M_3} + \int_d^s q_2(\sigma)d\sigma\right)ds < \infty. \tag{5.7}
\]

From (5.5), (5.6), (5.7), and Theorem 1.3.12 we have

\[
\|Z_2x_n - Z_2x^*\| \\
= \sup_{b \leq t < \infty} \left| (Z_2x_n)(t) - (Z_2x^*)(t) \right| \\
= \sup_{b \leq t < \infty} \left| \int_d^t F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right)ds - \int_d^t F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right)ds \right| \\
\leq \sup_{b \leq t < \infty} \int_d^t \left| F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right|ds \\
\leq \int_d^\infty \left| F\left(s, -1 + \int_d^s G(\sigma, x_n(\sigma))d\sigma\right) - F\left(s, -1 + \int_d^s G(\sigma, x^*(\sigma))d\sigma\right) \right|ds \to 0,
\]

as \( n \to \infty \), which implies the continuity of \( Z_2 \) in \( X_2 \).

The proof of (3). Again, we will use Theorem 1.3.14 and Theorem 1.3.15 to prove that \( Z_2(X_2) \) is pre-compact in \( CB[d, \infty) \). What we need to do is to prove that for any sequence \( \{x_n\} \in X_2 \), the sequence \( \{Z_2x_n\} \) is uniformly bounded and equicontinuous on any
bounded closed subinterval $[b_1, b_2]$ of $[d, \infty)$. Because $x_n \in X_2$ and $Z_2(X_2) \subset X_2$, we have $-2 \leq (Z_2x_n)(t) \leq -1$ for all $t \in [d, \infty)$ and $n \in \mathbb{N}$, then $\{Z_2x_n\}$ is uniformly bounded.

To show the equicontinuity of $\{Z_2x_n\}$ on $[b_1, b_2]$, we need to prove that for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|(Z_2x_n)(t_1) - (Z_2x_n)(t_2)| < \epsilon$$

for any $n \in \mathbb{N}$ and $t_1, t_2 \in [b_1, b_2]$ such that $|t_1 - t_2| < \delta$.

Note that

$$(Z_2x_n)'(t) = F(s, -1 + \int_d^t G(s, x_n(s))ds)$$

$$\geq p_2(t)f_2\left(-1 + \int_d^t q_2(t)g_2(x_n(s))ds\right)$$

$$\geq -Mf_2(M_3)\left(-p_2(t)f_2\left(-\frac{1}{M_3} + \int_d^t q_2(s)ds\right)\right)$$

$$= Mf_2(M_3)\left(p_2(t)f_2\left(-\frac{1}{M_3} + \int_d^t q_2(s)ds\right)\right).$$

By the Mean Value Theorem, there exists $\xi \in [t_1, t_2]$ such that

$$|(Z_2x_n)(t_1) - (Z_2x_n)(t_2)| = |(Z_2x_n)'(\xi)(t_1 - t_2)|$$

$$\leq -Mf_2(M_3)\max_{b_1 \leq t \leq b_2} \left(p_2(t)f_2\left(-\frac{1}{M_3} + \int_d^t q_2(s)ds\right)\right)|t_1 - t_2|.$$

To ensure $|(Z_2x_n)(t_1) - (Z_2x_n)(t_2)| < \epsilon$, we choose

$$\delta = \frac{\epsilon}{-Mf_2(M_3)\max_{b_1 \leq t \leq b_2} \left(p_2(t)f_2\left(-\frac{1}{M_3} + \int_d^t q_2(s)ds\right)\right)}$$

which implies the equicontinuity of $\{Z_2x_n\}$ on $[b_1, b_2]$. 
Theorem 1.3.13 guarantees that there exists a solution \( \bar{x} \in X_2 \) of \( Z_2 \) such that \( Z_2 \bar{x} = \bar{x} \).

Let

\[
\bar{y}(t) = -1 + \int_{d}^{t} G(s, \bar{x}(s)) ds.
\]

We can prove that \((\bar{x}, \bar{y})\) is a subclass \( B_b \) solution of system (1.1). Indeed, \( Z_2 \bar{x} = \bar{x} \) is the same as

\[
\bar{x}(t) = -1 + \int_{d}^{t} F\left(s, -1 + \int_{d}^{s} G(\sigma, \bar{x}(\sigma)) d\sigma\right) ds.
\]

Differentiating both sides implies that

\[
\bar{x}'(t) = F\left(t, -1 + \int_{d}^{t} G(s, \bar{x}(s)) ds\right)
= F(t, \bar{y}(t)).
\]

Also,

\[
\bar{y}'(t) = G(t, \bar{x}(t)).
\]

Therefore, \((\bar{x}, \bar{y})\) is a \( B_b \) solution of (1.1). \( \square \)
REFERENCES


