A PRELUDE AND FUGUE ON A THEME OF FACTORIZATION

by

Marissa Emiko North

An Abstract
presented in partial fulfillment
of the requirements for the degree of
Master of Science
in the Department of Mathematics and Computer Science
University of Central Missouri

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The past decade has seen much research on factorizations of elements in multiplicative monoids and integral domains. In particular, the use of graphical representations such as irreducible divisor graphs, compressed irreducible divisor graphs, and irreducible divisor simplicial complexes has yielded many results. This thesis continues this research by introducing the compressed irreducible divisor simplicial complex. We show that many results obtained by using the aforementioned constructions will also hold in the compressed irreducible divisor simplicial complex. Additionally, we obtain new results by comparing all of these constructions and expand on some ideas already existing in the literature. Lastly, we characterize the prime elements in a monoid by considering irreducible divisor graphs and irreducible divisor simplicial complexes.
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Chapter 1

Prelude

Factorization has always been both a difficult and curious thing to study. The first major contribution to the theory of factorization was the proof of the Fundamental Theorem of Arithmetic given by Euclid in 300 B.C. This theorem states that every positive integer can be expressed uniquely (up to order of the factors) as a product of primes. An example of this unique factorization is given by the following:

\[ 180 = 2^2 \cdot 3^2 \cdot 5 = 3^2 \cdot 2^2 \cdot 5. \]

These two factorizations of 180 are considered to be the same since only the order of terms in the factorization has changed.

However, as mathematicians have discovered over the past two centuries, unique factorization does not occur in every algebraic object. An example of an object over which factorization is not unique is the ring

\[ \mathbb{Z}[\sqrt{-5}] = \{ a + b\sqrt{-5} : a, b \in \mathbb{Z} \}. \]
Consider 6 in $\mathbb{Z}[\sqrt{-5}]$. Then 6 factors as:

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$

where $2, 3, 1 + \sqrt{-5},$ and $1 - \sqrt{-5}$ are pairwise nonassociate and irreducible. Thus 6 has two distinct factorizations in this ring.

The topic of unique and non-unique factorization has been thoroughly studied over the past 50 years. The reader who wants to delve further into this theory may wish to consult Geroldinger and Halter-Koch’s text \[GHK06\], a comprehensive overview of the present state of non-unique factorization theory.

In an attempt to better understand factorization in certain rings, Coykendall and Maney introduced the irreducible divisor graph in \[CM07\]. In this paper they give a way to graphically represent the factorizations of an element $x$ in an integral domain. The vertices in this graph are irreducible divisors of $x$ from a pre-chosen set of associate classes, and there exists an edge between two such vertices if and only if the corresponding irreducible divisors appear together in a factorization of $x$. (This idea of connecting these two branches of mathematics, Graph Theory and Commutative Ring Theory, was first shown to be of interest in \[Bec88\] by Istvan Beck.) Irreducible divisor graphs were utilized in \[CM07\] to characterize certain classes of domains including unique factorization domains (UFDs), finite factorization domains (FFDs), and half factorial domains (HFDs). For instance, it was shown that an integral domain $D$ is a UFD if and only if the irreducible divisor graph of each element $x$ in the integral domain $D$ forms a complete graph; meaning every vertex is connected to every other vertex in the graph by an edge.

Although useful in many regards, the sometimes over abundance of vertices and edges in irreducible divisor graphs make it difficult to view factorizations or notice other characterizing features. In \[ABS11\], the compressed irreducible divisor graph was introduced to
help simplify the irreducible divisor graph and to derive new results about factorizations of elements in integral domains. This new type of graph reduces the vertex set by use of compression. Compression, in this context, is defined as follows: given an element $x$ in an integral domain $D$, two irreducible divisors $a$ and $b$ of $x$ are considered to represent a single vertex in the compressed irreducible divisor graph if whenever $a$ appears in a factorization of $x$, $b$ does as well and vice versa. In [ABS11], properties of the compressed irreducible divisor graph are studied.

Like all things, the irreducible divisor and compressed irreducible divisor graph have certain drawbacks, especially when trying to determine the factorizations of an element from its graph. In an attempt to find a graphical construction that carries more information about individual factorizations of elements, the irreducible divisor simplicial complex was introduced in [BH13]. These irreducible divisor simplicial complexes provide a higher dimensional construction analogous to the irreducible divisor graphs, and do in fact, provide more detailed information about factorizations.

In this thesis we study each of the aforementioned constructions as well as the compressed irreducible divisor simplicial complex that we will soon introduce. In Chapter 1 we introduce terminology we use throughout the thesis. This includes algebraic and graph-theoretical definitions as well as an overview of reduced multiplicative monoids. In Chapter 2 we give a survey of irreducible divisor graphs, compressed irreducible divisor graphs, and irreducible divisor simplicial complexes. Also in this chapter we state and prove known results about these constructions but in a slightly more general setting. In Chapter 3 we introduce the compressed irreducible divisor simplicial complex and compare and contrast this construction with each of the previously mentioned constructions. We give some sufficient conditions for when they are isomorphic as graphs or simplicial complexes. In addition, we provide some new results about factorizations in terms of the compressed irreducible divisor simplicial complex. We end this thesis with Chapter 4 in which we give a result characterizing primes
in terms of the irreducible divisor graph and the irreducible divisor simplicial complex, coming full circle as we return to the uniqueness of prime factorization studied by Euclid in 300 B.C.

1.1 Algebraic Definitions

In this section we give some standard algebraic definitions which will be needed in future sections. We refer the reader to [DF04] for additional terminology and examples.

**Definition 1.1.1.** A ring $R$ is a set together with two binary operations, usually addition (+) and multiplication ($\cdot$), such that $(R, +)$ is an abelian group, multiplication is associative, and multiplication distributes over addition.

**Definition 1.1.2.** Let $R$ be a ring.

1. $R$ is **commutative** if multiplication is commutative.

2. $R$ has **identity** if it contains an element $1$, called the multiplicative identity, with the property that $1 \cdot r = r \cdot 1$ for all $r \in R$.

**Definition 1.1.3.** Let $D$ be a commutative ring with identity.

1. A nonzero element $a \in D$ is a **zero divisor** if there exists a nonzero element $b \in D$ such that $ab = 0$.

2. An element $u \in D$ is said to be a **unit** if there exist $v \in D$ such that $uv = vu = 1$.

3. $D$ is an **integral domain** if it contains no zero divisors.
**Definition 1.1.4.** Let $D$ be an integral domain.

1. A nonzero nonunit $r \in D$ is an **irreducible** element if whenever $r = ab$ with $a, b \in D$ either $a$ or $b$ is a unit in $D$.

2. A nonzero nonunit element $p \in D$ is called **prime** if whenever $p \mid ab$, then either $p \mid a$ or $p \mid b$.

3. Two elements $a$ and $b$ in $D$ are called **associates** if $a = ub$ where $u$ is some unit in $D$.

4. An element $r \in D$ is **square-free** if it is not divisible by a non-trivial square; that is, for any nonzero nonunit $s \in D$, $s^2 \nmid r$.

5. We denote the nonzero elements in $D$ by $D^*$ and $U(D)$ to be the units of $D$. Thus $D^* \setminus U(D)$ denotes the nonzero nonunit elements in $D$.

6. We define $\text{Irr}(D)$ to be the set of all irreducibles in $D$ and $\overline{\text{Irr}}(D)$ to be a prechosen set of associate class representatives, one representative from each class of nonzero irreducible associates.

**Definition 1.1.5.** An integral domain is **atomic** if every nonzero nonunit element can be factored into a (finite) product of irreducible elements.

**Definition 1.1.6.** Let $D$ be an atomic domain.

1. $D$ is a **finite factorization domain** (FFD) if every nonzero nonunit has only finitely many distinct nonassociate irreducible divisors.

2. $D$ is a **half factorial domain** (HFD) if for every $x \in D^* \setminus U(D)$ the number of terms in each factorization of $x$ into irreducibles is constant.

3. $D$ is a **bounded factorization domain** (BFD) if for each $x \in D^* \setminus U(D)$ there is a finite bound on the length of factorizations of $x$ into products of irreducible elements.
4. \( D \) is a **Unique Factorization Domain (UFD)** if for each nonzero nonunit element \( r \in D \), \( r \) has a unique factorization as a product of irreducible elements up to associates and permutations of the irreducible factors.

**Definition 1.1.7.** Let \( x \) be a nonzero nonunit in an atomic domain \( D \). Then

\[
L(x) = \{t : x = a_1 \cdots a_t, \text{ where each } a_i \text{ is irreducible}\}
\]

is the set of lengths of an element \( x \) in a BFD \( D \). The **elasticity** of an element \( x \in D \), denoted \( \rho(x) \), is

\[
\rho(x) = \frac{\max L(x)}{\min L(x)}
\]

and is a measure of how far the element \( x \) is from having unique factorization.

### 1.2 Examples of Algebraic Definitions

To illustrate some of the definitions in Section 1.2.1, consider the set

\[
\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\},
\]

which we use for many examples throughout this thesis. This set is a ring with addition defined by

\[
(a + b\sqrt{-5}) + (c + d\sqrt{-5}) = (a + c) + (b + d)\sqrt{-5}
\]

and multiplication defined by

\[
(a + b\sqrt{-5})(c + d\sqrt{-5}) = (ac - 5bd) + (bc + ad)\sqrt{-5}
\]

\[
= (ca - 5db) + (cb + da)\sqrt{-5}.
\]
Clearly multiplication is commutative making $\mathbb{Z}[\sqrt{-5}]$ a commutative ring. The ring $\mathbb{Z}[\sqrt{-5}]$ has identity since 

$$(a + b\sqrt{-5})(1 + 0\sqrt{-5}) = a + b\sqrt{-5}$$

for all $a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$. Moreover $(a + b\sqrt{-5})(c + d\sqrt{-5}) = 0$ if and only if $a = b = 0$ or $c = d = 0$. Thus $\mathbb{Z}[\sqrt{-5}]$ is a commutative ring that has identity and contains no zero divisors, so $\mathbb{Z}[\sqrt{-5}]$ is an integral domain.

The norm of $\mathbb{Z}[\sqrt{-5}]$ is the function $N : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}$ defined by 

$$N(a + b\sqrt{-5}) = a^2 + 5b^2.$$ 

One easily checks that 

$$N((a + b\sqrt{-5})(c + d\sqrt{-5})) = N(a + b\sqrt{-5})N(c + d\sqrt{-5})$$

for all $a + b\sqrt{-5}, c + d\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$.

Using the fact that $N$ is multiplicative, we note that $\alpha$ is a unit in $\mathbb{Z}[\sqrt{-5}]$ if and only if $N(\alpha) = 1$ which means $\alpha = 1$ or $\alpha = -1$. Also, since $N$ is multiplicative, we can check that $1 + \sqrt{-5}$ is an irreducible element in $\mathbb{Z}[\sqrt{-5}]$. Suppose 

$$1 + \sqrt{-5} = \alpha \beta$$

for some $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$. Then 

$$6 = N(\alpha)N(\beta).$$

One checks that $N(\alpha) \in \{2, 3\}$ is impossible and so $N(\alpha) = 1$ and $N(\beta) = 6$ or vice versa. Thus $1 + \sqrt{-5}$ is an irreducible element. Using a similar process, we can show $1 - \sqrt{-5}, 2,$ and $3$ are all irreducible elements in $\mathbb{Z}[\sqrt{-5}]$. Moreover, since $1$ and $-1$ are the only units,
these four elements are pairwise nonassociates. Therefore

\[ 6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \]

gives two distinct factorizations of 6 in \( \mathbb{Z}[\sqrt{-5}] \). Furthermore, one can check that these are the only factorizations of 6 since \( N(\alpha) = 36 \) factors only as \( 36 = 2^2 3^2 \) in the natural numbers and hence \( \rho(x) = \frac{2}{2} = 1 \).

Lastly, we show that 2 is not prime in \( \mathbb{Z}\sqrt{-5} \). Assume 2 is prime in \( \mathbb{Z}[\sqrt{-5}] \). Since 2 is irreducible and from the above we see that 2 divides \((1 + \sqrt{-5})(1 - \sqrt{-5})\) but does not divide either term, 2 is not prime in \( \mathbb{Z}[\sqrt{-5}] \). We have found an irreducible element that is not prime, and thus the ring \( \mathbb{Z}[\sqrt{-5}] \) is not a UFD.

### 1.3 Reduced Multiplicative Monoids

When studying factorizations in an integral domain, one only needs to consider multiplication of elements since addition plays at most a minor role. Moreover, since we do not care to distinguish factorizations which differ only by associativity of their factors, the study of factorization in an integral domain \( D \) becomes the study of factorization in the reduced multiplicative monoid \( D^*/\sim \), denoted by \( D^\bullet \), where \( a \sim b \) if and only if \( a \) and \( b \) are associates.

**Definition 1.3.1.** A reduced cancellative multiplicative monoid is a set \( H \) together with a binary operation \( \cdot \) with identity 1 such that

1. If \( ab = 1 \), then \( a = b = 1 \) for all \( a, b \in H \).

2. If \( ab = ac \) for all \( a, b, c \in H \), then \( b = c \).
**Definition 1.3.2.** Let $H$ be a reduced cancellative multiplicative monoid.

1. An irreducible element in $H$ is an element $a$ with the property that if $a = bc$ with $b, c \in H$, then either $b = 1$ or $c = 1$.

2. An element $p \in H$ is prime if whenever $p | ab$, then either $p | a$ or $p | b$.

3. We define $\text{Irr}(H)$ to be the set of irreducible elements in $H$.

**Remark 1.3.3.** For the rest of this thesis, we assume every monoid is reduced and cancellative.

Note that an integral domain $D$ is a UFD if and only if the monoid $D^*/\sim$ is free.

**Definition 1.3.4.** Let $H$ be a reduced multiplicative monoid. $H$ is said to be free if for every $x \in H\backslash\{1\}$, $x$ can be expressed uniquely as a product

$$x = a_1^{n_1} \cdots a_m^{n_m}$$

where each $a_i$ is an irreducible element of $H$ and $n_i \in \mathbb{Z}^+$ for all $i \in \{1, \ldots, m\}$.

We now recall the following useful theorem [GHK06, Theorem 1.1.10.2] which characterizes free monoids.

**Theorem 1.3.5.** Let $H$ be a monoid.

1. Every prime element of $H$ is irreducible.

2. If $H$ is atomic, then $H$ is free if and only if every irreducible element is prime.

We now give definitions for types of monoids analogous to the types of rings defined in Definitions 1.1.5 and 1.1.6.
Definition 1.3.6. A monoid \( H \) is **atomic** if for every \( x \in H \setminus \{1\} \), \( x \) can be factored into a (finite) product of irreducible elements.

Definition 1.3.7. Let \( H \) be an atomic monoid.

1. \( H \) is a **finite factorization monoid** (FFM) if for every \( x \in H \setminus \{1\} \), \( x \) has only finitely many distinct irreducible divisors.

2. \( H \) is a **half factorial monoid** (HFM) if for every \( x \in H \setminus \{1\} \) the number of terms in each factorization of \( x \) into irreducibles is constant.

3. \( H \) is a **bounded factorization monoid** (BFM) if for each \( x \in H \setminus \{1\} \) there is a finite bound on the length of factorizations of \( x \) into products of irreducible elements.

Definition 1.3.8. Let \( x \) be a nonunit of an atomic monoid \( H \). Then

\[
L(x) = \{ t : x = a_1 \cdots a_t \text{ where each } a_i \text{ is irreducible} \}
\]

is the set of lengths of an element \( x \) in a BFM \( H \). The **elasticity** of an element \( x \in H \), denoted \( \rho(x) \), is 

\[
\rho(x) = \frac{\max L(x)}{\min L(x)}.
\]

1.4 Example of a Reduced Multiplicative Monoid

In addition to the reduced multiplicative monoid of nonzero elements of an integral domain, there are many interesting reduced multiplicative monoids. We give one here which occurs often in the literature and which will provide an example of various properties in later sections. We refer the reader to [GHK06] for additional information about this particular reduced multiplicative monoid.
Definition 1.4.1. Let $G$ be a finite abelian group with binary operation given by addition.

1. The **free abelian monoid** with basis $G$, denoted by $\mathcal{F}(G)$, is defined by

$$\mathcal{F}(G) = \{g_1^{n_1} \cdots g_k^{n_k} : g_i \in G \text{ and } n_i \in \mathbb{N}\}$$

with operation given by concatenation.

2. Define a monoid homomorphism

$$\phi : \mathcal{F}(G) \rightarrow G$$

by $\phi(\alpha) = n_1 + \cdots + n_k$ where $\alpha = g_1^{n_1} \cdots g_k^{n_k}$. An element $\alpha \in \mathcal{F}(G)$ is a **zero-sum sequence** if and only if $\phi(\alpha) = 0$ in $G$.

3. We call $\alpha$ a **minimal zero-sum sequence** if there does not exist a proper subsequence of $\alpha$ which is also a zero-sum sequence.

We now can define a block monoid.

Definition 1.4.2. The **block monoid** of $G$, denoted $\mathcal{B}(G)$, is defined by

$$\mathcal{B}(G) = \{\alpha \in \mathcal{F}(G) : \phi(\alpha) = 0\}.$$  

The irreducible elements of $\mathcal{B}(G)$ are the minimal zero-sum sequences.

We now give a concrete example.
Example 1.4.3. Consider the block monoid \( B(\mathbb{Z}/4\mathbb{Z}) \) consisting of strings of elements \( g_1 \cdots g_t \) where each \( g_i \) is in the group \( \mathbb{Z}/4\mathbb{Z} = \{0, 1, 2, 3\} \) and \( g_1 + \cdots + g_t = 0 \) in \( \mathbb{Z}/4\mathbb{Z} \). One can easily check that the irreducible elements in this monoid are:

\[ 13, 1^4, 3^4, 23^2, 1^22, 0, \text{ and } 2^2. \]

Then the element \( x = 01^22^23^2 \) in \( \mathbb{Z}/4\mathbb{Z} \) has only the following factorizations into irreducibles:

\[ x = 0(13)^2(2^2) = 0(1^22)(23^2). \]

Since the factorizations of \( x \) have lengths 3 and 4, \( L(x) = \{3, 4\} \) and \( \rho(x) = \frac{4}{3} \).

1.5 Graphs

In this section we provide the basic graph-theoretical terminology we will need in later sections. We refer the reader to [CZ05] for additional terminology and examples.

Definition 1.5.1. A graph, denoted \( G = (V, E) \), is an ordered pair where \( V \) is the nonempty set of vertices in \( G \) and \( E \) the set of edges in \( G \) whose elements are subsets of \( V \) of cardinality 2.

Definition 1.5.2. Let \( G = (V, E) \) be a graph.

1. There is a loop on a vertex \( a \in V \) if there exists a “pseudo-edge” from \( a \) to \( a \); meaning a multiset \( \{a, a\} \) of \( V \).

2. We say \( u, v \in V \) are adjacent in \( G \) if \( \{u, v\} \in E \).

3. The order of \( G \) is the number of vertices in \( G \) if finite.
4. A walk between two vertices \( u \) and \( v \) in \( G \) is a sequence of edges in \( G \), denoted

\[
\{u, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}, \{a_n, v\}.
\]

5. \( G \) is connected if for every pair of vertices \( u \) and \( v \), there is a walk from \( u \) to \( v \) in \( G \).

6. \( G \) is complete if every distinct pair of vertices in \( G \) are adjacent. A complete graph of order \( n \) is denoted by \( K_n \).

To illustrate these definitions, consider the following examples.

**Example 1.5.3.** Let \( G = (V, E) \) where \( V = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} \) and

\[
E = \{\{a_1, a_2\}, \{a_1, a_6\}, \{a_2, a_3\}, \{a_6, a_3\}, \{a_3, a_4\}, \{a_3, a_7\}, \{a_4, a_5\}, \{a_5, a_7\}\}.
\]

A visual representation of \( G \) is given in Figure 1.1.

![Figure 1.1: Example of a Connected Graph](image-url)

The order of \( G \) is 7. The sequence of edges \( \{a_1, a_2\}, \{a_2, a_3\}, \{a_3, a_4\}, \{a_4, a_5\} \) is a walk from \( a_1 \) to \( a_5 \). Note that \( G \) is connected since between every distinct pair of vertices \( a_i, a_j \in V \), there is a walk from \( a_i \) to \( a_j \). However, this graph is not complete since \( \{a_6, a_7\} \notin E \) and hence there is no edge between \( a_6 \) and \( a_7 \).
Example 1.5.4. Let \( G = (V, E) \) where \( V = \{a_1, a_2, a_3\} \) and

\[
E = \{\{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}\}.
\]

A visual representation of \( G \) is given in Figure 1.2.

![Figure 1.2: Example of a Complete Graph](image)

The order of \( G \) is 3. Note that \( G \) is a complete graph since for every distinct pair of vertices \( a_i, a_j \in V, \{a_i, a_j\} \in E \). We denote this graph by \( K_3 \). Clearly every complete graph is connected.

1.6 Simplicial Complexes

In this section we generalize graphs to higher dimensions by introducing simplicial complexes.

Definition 1.6.1. A simplicial complex, denoted \( S = (V, F) \), is an ordered pair where \( V \) is a nonempty set of vertices and \( F \) is a set of faces (subsets of \( V \)) in \( S \) that satisfy the following properties:

1. \( \{v\} \in F \) for all \( v \in V \).

2. If \( B \in F \) and \( A \subset B \), then \( A \in F \).

Definition 1.6.2. Let \( S = (V, F) \) be a simplicial complex.

1. The dimension of a face \( A \in F \) of finite cardinality is \( \dim A = |A| - 1 \).
2. The maximal faces of $S$ with respect to containment are called **facets**.

3. The $n$-skeleton, $k_n(S)$, of $S$ consists of all faces of dimension $n$ and lower for some $n \in \mathbb{Z}^+$.

We illustrate these definitions with the following example.

**Example 1.6.3.** Let $S = (V,F)$ where $V = \{a, b, c, d\}$ and

$$F = \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}, \{a, b, c\}.$$ 

A visual representation of $S$ is given in Figure 1.3.

![Figure 1.3: Example of a Simplicial Complex](image-url)

The 0-dimensional faces are the singleton sets $\{a\}, \{b\}, \{c\}$, and $\{d\}$ which we graphically represent by points. The 1-dimensional faces are $\{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}$ and are graphically represented by line segments. The 2-dimension face is $\{a, b, c\}$ and is represented graphically by a shaded triangle. A 3-dimensional face, which this particular complex does not have, would be graphically represented by a solid tetrahedra. It is not possible to graphically represent a 4-dimensional or higher face so it is left to the reader’s imagination.

Consider $k_1(S) = (V,F)$ where $V = \{a, b, c, d\}$ and

$$F = \emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{c, d\}.$$
The 1–skeleton is given in Figure 1.4.

![Figure 1.4: 1-skeleton of S](image)

Note that if $S$ is a simplicial complex, then $k_1(S)$ is a graph consisting only of faces of dimension of at most one.
Chapter 2

Exposition

2.1 Irreducible Divisor Graphs

The irreducible divisor graph introduced by Coykendall and Maney in [CM07] allows us to visually see a graphical representation of the factorizations of an element $x$ in an atomic monoid $H$.

In this section we formally define the irreducible divisor graph. We then give some examples and discuss how these graphs can be used to understand factorizations in certain algebraic structures as well as their shortcomings. Also, we prove known results about irreducible divisor graphs of elements in atomic domains in the more general setting of atomic monoids.

We now give the formal definition of an irreducible divisor graph for an element in an atomic monoid, a definition parallel to the one given in [CM07] for an irreducible divisor graph of an element in an atomic domain.
Definition 2.1.1. Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. The **Irreducible Divisor Graph** of $x$, denoted $G_H(x)$, is given by $(V, E)$, where

$$V = \{y \in \text{Irr}(H) : y \mid x\}$$

is the vertex set of $G(x)$ and if $y_1, y_2 \in V$, then there is an edge $\{y_1, y_2\} \in E$, the edge set in $G(x)$, if and only if $y_1y_2 \mid x$.

If $y \in V$ with $y^n \mid x$ but $y^{n+1} \nmid x$, then we place $n - 1$ loops on the vertex $y$. When the atomic monoid is clear from context, we write $G(x)$ in place of $G_H(x)$.

To illustrate this definition consider the following examples.

Example 2.1.2. Let $H = \mathbb{Z}[\sqrt{-5}]^*$ and consider $x = 18$. Then the only factorizations of 18 are

$$18 = 2 \cdot 3^2 = 2(2 + \sqrt{-5})(2 - \sqrt{-5}) = 3(1 + \sqrt{-5})(1 - \sqrt{-5}).$$

For convenience, let $\alpha = 1 + \sqrt{-5}$ and $\beta = 2 + \sqrt{-5}$, and denote their complex conjugates as $\overline{\alpha} = 1 - \sqrt{-5}$ and $\overline{\beta} = 2 - \sqrt{-5}$. The irreducible divisor graph of $G(18)$ is given in Figure 2.1.

![Figure 2.1: $G(18)$ in $\mathbb{Z}[\sqrt{-5}]^*$](image-url)
From the factorizations of 18, we see that the vertex and edge set of $G(18)$ are

$$V = \{2, 3, \alpha, \alpha, \beta, \beta\},$$

$$E = \{\{2, 3\}, \{2, \beta\}, \{2, \beta\}, \{3, \alpha\}, \{3, \alpha\}, \{\beta, \beta\}, \{\alpha, \alpha\}\},$$

and that there is a single loop on the vertex 3.

The objective of studying the irreducible divisor graph of an element in an atomic monoid is to be able to determine information about the factorizations of this element and perhaps draw some conclusions about the monoid in which the element lives. If we delete the labels on the vertices in Figure 2.1 and replace them by arbitrary labels, we obtain the graph seen in Figure 2.2.

When looking at Figure 2.2 and thinking of this as an irreducible divisor graph of an unknown element $x \in H\backslash\{1\}$ where $H$ is an atomic monoid, observe that $x$ has a factorization of the form $x = a_1a_2a_3$. To see this, suppose $x = a_1a_2$ and $x = a_1a_3$, setting these factorizations equal to each other

$$a_1a_2 = a_1a_3.$$ 

By cancellation in $H$ we have, $a_2 = a_3$, a contradiction. Thus $x = a_1a_2a_3$. By definition of $G(x)$, there exists a factorization of $x$ of the form $x = a_1a_4^{n_1}$. However, we cannot confirm whether $n_1 = 1$ or $n_1 = 2$. A similar problem occurs when considering a factorization of $x$. 

![Figure 2.2: G(18) with Arbitrary Labels](image-url)
using the vertices $a_4, a_5,$ and $a_6$. Without further information we cannot tell if

$$x = a_4^2 a_5 a_6, x = a_4 a_5 a_6, x = a_4^2 a_6, x = a_4 a_6, x = a_4 a_5, \text{ or } x = a_5 a_6$$

is an actual factorization of $x$. Notice that the source of this undecidability lies in the existence of the loop on the vertex $a_4$.

**Example 2.1.3.** Let $H = \mathbb{Z}[\sqrt{-5}]^\ast$ and consider $x = 108$. Then the only factorizations of $108$ are

$$108 = 2^2 3^3 = 23^2 \alpha \overline{\alpha} = 2^2 3_3 \beta \overline{\beta} = 3 \alpha^2 \overline{\alpha}^2 = 2 \alpha \overline{\alpha} \beta \overline{\beta}.$$  

The irreducible divisor graph of $G(108)$ is given in Figure 2.3.

![Figure 2.3: G(108) in \(\mathbb{Z}[\sqrt{-5}]^\ast\)](image-url)
From the factorizations of 108 we see that the vertex set and edge set of $G(108)$ are

$$V = \{2, 3, \alpha, \bar{\alpha}, \beta, \bar{\beta}\},$$

$$E = \{\{2, 3\}, \{2, \alpha\}, \{2, \bar{\alpha}\}, \{2, \beta\}, \{2, \bar{\beta}\}, \{3, \alpha\}, \{3, \bar{\alpha}\}, \{3, \beta\}, \{3, \bar{\beta}\}, \{\alpha, \alpha\}, \{\beta, \beta\}, \{\alpha, \beta\}, \{\alpha, \bar{\beta}\}, \{\beta, \bar{\alpha}\}, \{\beta, \bar{\beta}\}\},$$

there is a double loop on vertex 3, and a single loop on vertices 2, $\alpha$, and $\bar{\alpha}$.

Although $G(108)$ is a complete graph, there is no factorization of 108 involving all of the irreducible divisors of 108.

We now state and prove the main result of [CM07] in which Coykendall and Maney were able to characterize a UFD in terms of the irreducible divisor graph. We give Theorem 2.1.4 in the slightly more general framework of the atomic monoid. The proof here is similar to the proof given in [ABS13].

**Theorem 2.1.4.** Let $H$ be an atomic monoid. The following are equivalent:

1. $H$ is a free monoid.

2. $G(x)$ is complete for every $x \in H \setminus \{1\}$.

3. $G(x)$ is connected for every $x \in H \setminus \{1\}$.

**Proof.** (1) $\Rightarrow$ (2) Suppose $H$ is a free monoid. Then, by definition, $x$ can be written uniquely as $x = a_1^{n_1} \cdots a_m^{n_m}$ where each $a_i$ is an irreducible element in $H$ and $n_i \in \mathbb{Z}^+$, for all $i \in \{1, \ldots, m\}$. Note that the only factorization of $x$ is given above. Thus $V(G(x)) = \{a_1, \ldots, a_m\}$, and for each pair of distinct $a_i, a_j \in \{a_1, \ldots, a_m\}, a_i a_j \mid x$. Therefore $\{a_i, a_j\} \in E(G(x))$ for all $i, j \in \{1, \ldots, m\}$ where $i \neq j$, and $G(x)$ is complete for every $x \in H \setminus \{1\}$. 


(2) $\Rightarrow$ (3) Suppose $G(x)$ is complete for every $x \in H \backslash \{1\}$. Then every distinct pair of vertices in $G$ are adjacent and thus every pair of vertices in $G$ are connected by some walk. Therefore $G(x)$ is connected for every $x \in H \backslash \{1\}$.

(3) $\Rightarrow$ (1) Suppose $G(x)$ is connected for every $x \in H \backslash \{1\}$. Let $S = \{z \in H \backslash \{1\} : z$ has at least two distinct factorizations$\}$, and let $m = \min_{z \in S} \{k : z = \pi_1 \cdots \pi_k, \pi_i \in \text{Irr}(H)$ for $i \in \{1, \ldots, k\}\}$. Since $z$ is not an irreducible element of $H, m \geq 2$. Thus there exists $y \in S$ such that $y = \pi_1 \cdots \pi_m$ where each $\pi_i$ is an irreducible element. Since $y \in S$, it has another factorization, say $y = \alpha_1 \cdots \alpha_n$, where $n \geq m$. Then

$$y = \pi_1 \cdots \pi_m = \alpha_1 \cdots \alpha_n.$$ 

Suppose $\pi_i = \alpha_j$ for some $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Then

$$\frac{y}{\pi_i} = \pi_1 \cdots \pi_{i-1} \pi_{i+1} \cdots \pi_m = \alpha_1 \cdots \alpha_{j-1} \alpha_{j+1} \cdots \alpha_n.$$ 

Note that $\frac{y}{\pi_i} \in S$ and has a factorization length of $m-1$, but this contradicts the minimality of $m$. Thus $\pi_i \neq \alpha_j$ for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Since $G(x)$ is connected for every $x \in H \backslash \{1\}, G(y)$ is connected. Thus there exists an edge between some $\pi_i$ and some $\alpha_j$. Without loss of generality, assume $\{\pi_1, \alpha_1\} \in E(G(y))$. Then there exists another factorization of $y$ given by

$$y = \pi_1 \alpha_1 \beta_1 \cdots \beta_l.$$
where \( l \geq m - 2 \). Setting these two factorizations of \( y \) equal to each other, we obtain

\[
y = \pi_1 \alpha_1 \beta_1 \cdots \beta_l = \pi_1 \cdots \pi_m.
\]

Then, by cancellation,

\[
\frac{y}{\pi_1} = \alpha_1 \beta_1 \cdots \beta_l = \pi_2 \cdots \pi_m.
\]

Thus \( \frac{y}{\pi_1} \in S \) with a factorization length of \( m - 1 \), again contradicting the minimality of \( m \). Therefore, \( S \) is empty and \( H \) is free.

In Chapter 4 we state and prove a result, Theorem 4.0.2, characterizing prime elements in an atomic monoid in terms of irreducible divisor graphs. Since, by Theorem 1.3.5, an atomic monoid is free if and only if every irreducible element is prime, Theorem 2.1.4 is then an immediate corollary.

The last result of this section gives a bound on the elasticity of an element \( x \) in a BFM \( H \). This result was first given in [ABS13] in the context of atomic domains. The proof given here is similar to the proof in [ABS13] but for a slightly more general setting. We will obtain an improvement on this bound in Proposition 2.3.8.

**Proposition 2.1.5.** Let \( x \) be a nonirreducible nonunit of a BFM \( H \). Choose a complete subgraph of \( G(x) \) with vertex set \( V = \{a_1, \ldots, a_t\} \) such that \( t + l \) is maximal where \( l \) is the number of loops on the \( t \) vertices in \( V \). Then

\[
\rho(x) \leq \frac{t + l}{2}.
\]

**Proof.** Since \( x \) is a nonirreducible element, \( \min L(x) \geq 2 \). Now to show the claimed bound, we only need to show that \( \max L(x) \leq t + l \). Let \( M = \max L(x) \). Then there exists a
factorization of $x$ given by:

$$x = b_1^{n_1} \cdots b_s^{n_s}$$

where each $b_i$ is an irreducible element, $b_i \neq b_j$ unless $i = j$, and $\sum_{i=1}^{s} n_i = M$. Then each $b_i$ is a vertex in $G(x)$ and by definition, $\{b_i, b_j\} \in E(G(x))$ for all $i, j \in \{1, \ldots, s\}$, where $i \neq j$. By definition, $G(x)$ has a complete subgraph with vertex set $\{b_1, \ldots, b_s\}$ having a total of at least $l_s = \sum_{i=1}^{s} (n_i - 1)$ loops on the $s$ vertices. Therefore taking the maximum over all complete subgraphs of $G(x)$,

$$\rho(x) \leq \frac{t + l}{2}.$$ 

\[\Box\]

### 2.2 Compressed Irreducible Divisor Graphs

The concept of compressed irreducible divisor graphs was introduced in [ABS13] in an attempt to better understand the relationship between factorizations in an atomic domain and graphs related to these factorizations. In this section we formally define the compressed irreducible divisor graph. Also, we state and prove results from [ABS13] but in a more general setting of atomic monoids. In particular, we compare the irreducible divisor graph with its corresponding compressed irreducible divisor graph.

We now give the definition of compression in terms of atomic monoids, which is parallel to the definition in [ABS13] for compression in atomic domains.

**Definition 2.2.1.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. We say $a, b \in \text{Irr}(H)$ are $x$-equivalent, denoted by $a \sim_x b$, if whenever $a$ appears in a factorization of $x$, $b$ does as well and vice versa.
Note that this is an equivalence relation on the set of irreducible divisors of an element $x$ in an atomic monoid $H$. If $a \in H \setminus \{1\}$ is an irreducible divisor of an element $x \in H \setminus \{1\}$, we denote
\[ [a]_x = \{ b : a \sim_x b \}. \]

Note that if $b \in [a]_x$, then $[a]_x = [b]_x$. When an element $x$ is clear from context, we write $[a]$ in place of $[a]_x$.

We are now able to define the compressed irreducible divisor graph.

**Definition 2.2.2.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. The compressed irreducible divisor graph of $x$, denoted $G_c(x)$, is given by $(V,E)$ where
\[ V = \{ [y]_x : y \in \text{ Irr}(H) \text{ and } y \mid x \} \]
is the vertex set of $G_c(x)$ and if $[y_1]_x, [y_2]_x \in V$, then $\{ [y_1]_x, [y_2]_x \} \in E$, the edge set of $G_c(x)$, if and only if $y_1y_2 \mid x$.

Note that in the compressed irreducible divisor graph, loops are not well-defined and thus we omit them. Indeed, if $a \neq b$, $[a]_x = [b]_x$, it would not be clear whether a loop on the vertex $[a]_x$ in the compressed irreducible divisor graph means $a^2 \mid x, b^2 \mid x, a^2b^2 \mid x, a^2b \mid x$, or $ab^2 \mid x$.

To illustrate this definition we reconsider Examples 2.1.2 and 2.1.3 from Section 2.1.

**Example 2.2.3.** Let $H = \mathbb{Z}[\sqrt{-5}]^\bullet$ and consider $x = 18$. When looking at the factorizations of 18 in Example 2.1.2, we see that $\alpha$ and $\overline{\alpha}$ always appear together and that $\beta$ and $\overline{\beta}$ always appear together in factorizations of 18. Thus $[\alpha] = [\overline{\alpha}]$ and $[\beta] = [\overline{\beta}]$. Hence $G_c(18)$ is given in Figure 2.4.
CHAPTER 2. EXPOSITION

Figure 2.4: $G_c(18)$ in $\mathbb{Z}[\sqrt{-5}]^\bullet$


Example 2.2.4. Let $H = \mathbb{Z}[\sqrt{-5}]^\bullet$ and consider $x = 108.$ Once again $\alpha$ and $\overline{\alpha}$ always appear together and $\beta$ and $\overline{\beta}$ always appear together in factorizations of 108. Thus $[\alpha] = [\overline{\alpha}]$ and $[\beta] = [\overline{\beta}].$ Hence $G_c(108)$ is given in Figure 2.5.

Figure 2.5: $G_c(108)$ in $\mathbb{Z}[\sqrt{-5}]^\bullet$

By comparing Figure 2.3 with Figure 2.5, we see that $G_c(108)$ is much simpler than $G(108).$

The following theorem gives an interesting connection between the irreducible divisor graph and the compressed irreducible divisor graph of an element and extends Theorem 3.6 in [ABS13]. Again, this result has already appeared in the above mentioned paper in terms of factorizations in an atomic domain; here we prove it in the more general setting of atomic monoids.
Theorem 2.2.5. Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Then:

1. $G(x)$ is connected if and only if $G_c(x)$ is connected.

2. $G(x)$ is complete if and only if $G_c(x)$ is complete.

3. $H$ is a free monoid if and only if $G_c(x) \cong K_1$ for all $x \in H \setminus \{1\}$.

Proof. Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Define $G(x) = (V, E)$ and $G_c(x) = (V', E')$.

(1) Consider $[a]_x, [b]_x \in V'$ where $[a]_x \neq [b]_x$. Then $a, b \in V$. Since $G(x)$ is connected, there exists a walk from $a$ to $b$ denoted by the sequence of edges

$$\{a, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}, \{c_n, b\}$$

in $G(x)$. Let $i = \min\{1, \ldots, n\}$ such that $[a]_x \neq [c_i]_x$ and $j = \max\{1, \ldots, n\}$ such that $[b]_x \neq [c_j]_x$. Then the sequence of edges

$$\{[a]_x, [c_i]_x\}, \{[c_i]_x, [c_{i+1}]_x\}, \ldots, \{[c_{j-1}]_x, [c_j]_x\}, \{[c_j]_x, [b]_x\}$$

is a walk in $G_c(x)$ where $\{[c_k]_x, [c_{k+1}]_x\}$ is omitted whenever $[c_k]_x = [c_{k+1}]_x$. Therefore $G_c(x)$ is connected.

Conversely, suppose $G_c(x)$ is connected. Consider $a, b \in V$. Suppose $[a]_x = [b]_x$. Then by definition, $ab \mid x$. Thus $\{a, b\} \in E$. Suppose $[a]_x \neq [b]_x$. Since $G_c(x)$ is connected, there exists a walk between $[a]_x$ and $[b]_x$ given by the sequence of edges

$$\{[a]_x, [c_1]_x\}, \{[c_1]_x, [c_2]_x\}, \ldots, \{[c_{n-1}]_x, [c_n]_x\}, \{[c_n]_x, [b]_x\}$$

where $[c_i]_x \in V'$ for all $i \in \{1, \ldots, n\}$.
Thus there exists a walk from $a$ to $b$ in $G(x)$, where we pick one element $c_i$ from each equivalence class $[c_i]_x$ to create this walk given by the sequence of edges

$$
\{a, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}, \{c_n, b\}.
$$

Therefore $G(x)$ is connected.

(2) Let $a, b \in V$ where $a \neq b$. Suppose $[a]_x = [b]_x$. Then by definition, $ab | x$ and thus $\{[a]_x, [b]_x\} \in E'$. Suppose $[a]_x \neq [b]_x$. Since $G(x)$ is complete, there exists an edge between $a$ and $b$ and thus $ab | x$. By definition, this means $\{[a]_x, [b]_x\} \in E'$. Thus $G_c(x)$ is complete.

Conversely, let $[a]_x, [b]_x \in V'$ where $[a]_x \neq [b]_x$. Then $a, b \in V$ where $a \neq b$. Since $G_c(x)$ is complete, $\{[a]_x, [b]_x\} \in E'$ which by definition means $ab | x$ where $a \in [a]_x$ and $b \in [b]_x$. Thus $\{a, b\} \in E$. Therefore $G(x)$ is complete.

(3) Suppose $H$ is free. Then each $x \in H \setminus \{1\}$ can be written uniquely as

$$
x = a_1^{n_1} \cdots a_m^{n_m}
$$

for $\{a_1, \ldots, a_m\} \in V$ and $n_i \geq 1$ for all $i \in \{1, \ldots, m\}$. Since this is the only factorization of $x$, we have $[a_1] = \cdots = [a_m]$, giving us $V' = \{[a]_x\}$ where $\{a_1, \ldots, a_m\} \in [a]_x$. Therefore $G_c(x) \cong K_1$ for every $x \in H \setminus \{1\}$.

Conversely, from the hypothesis, $G_c(x)$ consists of a single vertex for all $x \in H \setminus \{1\}$. Thus, by definition of compression, if $a_i$ appears in a factorization of $x$, then $a_j$ must always appear as well and vice versa for all $i, j \in \{1, \ldots, m\}$ where $i \neq j$. Since $x \in H \setminus \{1\}$ it has a factorization given by $x = a_1^{n_1} \cdots a_m^{n_m}$. For the sake of contradiction, suppose $x$ has two distinct factorizations, say

$$
x = a_1^{n_1} \cdots a_m^{n_m} = a_1^{k_1} \cdots a_m^{k_m}
$$

where for some $i, n_i \leq k_i$. Without loss of generality, suppose $n_1 < k_1$. Then we can divide
through by $a_1^{n_1}$ to get
\[
\frac{x}{a_1^{n_1}} = a_2^{n_2} \cdots a_m^{n_m} = a_1^{k_1-n_1} a_2^{k_2} \cdots a_m^{k_m}.
\]

Note that $\frac{x}{a_1^{n_1}} \in H \setminus \{1\}$ and that $[a_1]_{a_1^{n_1}} \neq [a_2]_{a_1^{n_1}}$. Therefore $G_c(\frac{x}{a_1^{n_1}}) \not\cong K_1$. Thus we have a contradiction. Therefore $n_i = k_i$ for all $i \in \{1, \ldots, m\}$. Hence $H$ is free.

\[\square\]

**Corollary 2.2.6.** Let $H$ be an atomic monoid. The following are equivalent:

1. $H$ is a free monoid.

2. $G_c(x) \cong K_1$ for every $x \in H \setminus \{1\}$.

3. $G_c(x)$ is complete for every $x \in H \setminus \{1\}$.

4. $G_c(x)$ is connected for every $x \in H \setminus \{1\}$.

**Proof.**

(1) $\Rightarrow$ (2) Suppose $H$ is free. Then by Theorem 2.2.5, $G_c(x) \cong K_1$ for every $x \in H \setminus \{1\}$.

(2) $\Rightarrow$ (3) Suppose $G_c(x) \cong K_1$. Then by Theorem 2.2.5, $H$ is free. By Theorem 2.1.4, $G(x)$ is complete for every $x \in H \setminus \{1\}$ and by Theorem 2.2.5, $G_c(x)$ is complete for every $x \in H \setminus \{1\}$.

(3) $\Rightarrow$ (4) Suppose $G_c(x)$ is complete for every $x \in H \setminus \{1\}$, then by Theorem 2.2.5, $G(x)$ is complete for every $x \in H \setminus \{1\}$. By Theorem 2.1.4, $G(x)$ is connected for every $x \in H \setminus \{1\}$, and thus $G_c(x)$ is connected for every $x \in H \setminus \{1\}$ by Theorem 2.2.5.

(4) $\Rightarrow$ (1) Suppose $G_c(x)$ is connected for every $x \in H \setminus \{1\}$. Then $G(x)$ is connected for every $x \in H \setminus \{1\}$ and thus $H$ is free by Theorem 2.1.4.

The last result of this section compares the irreducible divisor graph and the compressed irreducible divisor graph of an element $x$ in an atomic monoid and gives a sufficient condition.
for when \( G(x) = G_c(x) \). The proof given here is similar to the proof in [ABS11] but for a slightly more general setting.

**Remark 2.2.7.** In Proposition 2.2.8 we state that under certain hypothesis \( G(x) = G_c(x) \). Technically, this is not correct since the vertices of \( G(x) \) are the irreducible divisors of \( x \) and the vertices of \( G_c(x) \) are sets of irreducible divisors of \( x \). However, when there is no compression, meaning \([a]_x = \{a\}\) for all \( a \in \text{Irr}(x)\), we consider the natural graph isomorphism mapping \( V(G(x)) \) to \( V(G_c(x)) \) by \( a \to [a] \) as an equivalence and write \( G(x) = G_c(x) \).

**Proposition 2.2.8.** Let \( H \) be an atomic monoid and let \( x \in H \backslash \{1\} \). Suppose \( G(x) \not\cong K_2 \) and that \( G(x) \) contains no subgraph isomorphic to \( K_n \) for \( n \geq 3 \). If \( G(x) \) is connected, then \( G(x) = G_c(x) \).

**Proof.** Let \( G(x) = (V, E) \) and \( G_c(x) = (V', E') \). Assume \( V = \{v\} \). Then \( E = \emptyset \) and thus \( G(x) = G_c(x) \).

Now assume \( G(x) \) contains at least three vertices. For the sake of contradiction, suppose \( G(x) \not\equiv G_c(x) \). Then there exist distinct vertices \( a, b \in V \) such that \([a]_x = [b]_x \). By definition, whenever \( a \) appears in a factorization of \( x \), \( b \) must appear as well and vice versa. Since \( G(x) \) contains more than two vertices and \( G(x) \) is connected, there exists \( c \in G(x) \) where \( c \neq a \) and \( c \neq b \) such that either \( \{a, c\} \in E \) or \( \{b, c\} \in E(G(x)) \). Without loss of generality, suppose \( \{a, c\} \in E \). Then \( ac \mid x \). Since \( a \) appears in this factorization of \( x \), \( b \) must as well. Hence, \( abc \mid x \). Thus \( \{a, b\}, \{a, c\}, \{b, c\} \in E \). This is a contradiction since \( G(x) \) contains no subgraph isomorphic to \( K_3 \). Therefore \( G(x) = G_c(x) \).  

\( \square \)
2.3 Irreducible Divisor Simplicial Complexes

In Example 2.1.2 of Section 2.1 we saw that it is sometimes, but not always, possible to determine the distinct factorizations of an element \( x \) by viewing its irreducible divisor graph. In an attempt to better understand the relationship between factorizations and their graphical representations, the concept of the irreducible divisor simplicial complex was introduced in [BH13]. In this section we give the definition, illustrate the usefulness by way of a couple of examples, and give known results from this theory, but generalized from the setting of atomic domains to that of atomic monoids.

We now give the formal definition of an irreducible divisor simplicial complex for an element in an atomic monoid, a definition parallel to the one given in [BH13] of an irreducible divisor simplicial complex of an element in an atomic domain.

**Definition 2.3.1.** Let \( H \) be an atomic monoid and let \( x \in H \setminus \{1\} \). The **irreducible divisor simplicial complex** of \( x \), denoted \( S_H(x) \), is given by \((V,F)\), where

\[
V = \{y : y \in \text{Irr}(H) \text{ and } y \mid x\}
\]

is the vertex set of \( S_H(x) \) and with \( \{y_1, \ldots, y_n\} \in F \), the face set of \( S_H(x) \), if and only if \( y_1 \cdots y_n \mid x \).

Once again, if \( H \) is clear from context we write \( S(x) \) in place of \( S_H(x) \).

Again we refer to Examples 2.1.2 and 2.1.3 from Section 2.1 and illustrate these factorizations in terms of the irreducible divisor simplicial complex.

**Example 2.3.2.** Let \( H = \mathbb{Z}[\sqrt{-5}]^\ast \) and let \( x = 18 \). Recall from Example 2.1.2 the only factorizations of 18 are

\[
18 = 2 \cdot 3^2 = 2\beta\bar{\beta} = 3\alpha\bar{\alpha}.
\]

The irreducible divisor simplicial complex of \( S(18) \) if given in Figure 2.6.
Note that the vertex set $V$ of $S(18)$ is equal to the vertex set in $G(18)$. We note that the face set of $S(18)$ is $F = \emptyset \cup V \cup F_1 \cup F_2$, where

$$F_1 = \{\{2, 3\}, \{2, \beta\}, \{2, \overline{\beta}\}, \{3, \alpha\}, \{3, \overline{\alpha}\}, \{\alpha, \overline{\alpha}\}, \{\beta, \overline{\beta}\}\}$$

is the set of faces of dimension one and

$$F_2 = \{\{2, \beta, \overline{\beta}\}, \{3, \alpha, \overline{\alpha}\}\}$$

is the set of faces of dimension two. Note that $F_1$ is the edge set in $G(18)$ which we graphically represent by line segments, and $F_2$ is the set of faces of dimension two which we depict in the complex by shading triangles. The facets in $S(18)$ are:

$$\{2, 3\}, \{2, \beta, \overline{\beta}\}, \text{ and } \{3, \alpha, \overline{\alpha}\}.$$  

Notice that the facets in $S(18)$ correspond to the factorizations of $x$. This idea will be made clear in Propositions 2.3.5 and 2.3.6.

**Example 2.3.3.** Let $H = \mathbb{Z}[\sqrt{-5}]^*$ and let $x = 108$. Recall from Example 2.1.3 that the only factorizations of 108 are

$$108 = 2^2 3^3 = 2 \cdot 3^2 \alpha \overline{\alpha} = 2^2 3 \beta \overline{\beta} = 3 \alpha^2 \overline{\alpha}^2 = 2 \alpha \overline{\alpha} \beta \overline{\beta}.$$  

The irreducible divisor simplicial complex $S(108)$ is given in Figure 2.7.

Once again, the vertex set of $G(108)$ is the same as the vertex set in $S(108)$. We note that the face set of $S(108)$ is $F = \emptyset \cup V \cup F_1 \cup F_2 \cup F_3$. Once again, $F_1$ is the set of faces of dimension one. Note that $F_1$ is the edge set in $G(108)$.

$$F_2 = \{\{2,3,\alpha\}, \{2,3,\overline{\alpha}\}, \{2,\alpha,\overline{\alpha}\}, \{3,\alpha,\overline{\alpha}\}, \{2,3,\beta\}, \{2,3,\overline{\beta}\}, \{2,\beta,\overline{\beta}\}, \{3,\beta,\overline{\beta}\} , \{\alpha,\overline{\alpha},\beta\}, \{\alpha,\overline{\alpha},\overline{\beta}\}, \{\alpha,\beta,\overline{\beta}\}, \{\overline{\alpha},\beta,\overline{\beta}\}, \{2,\alpha,\beta\}, \{2,\alpha,\overline{\beta}\}, \{2,\beta,\overline{\beta}\}, \{2,\overline{\alpha},\overline{\beta}\}\}$$

is the set of faces of dimension two and is graphically represented by shaded triangles.

$$F_3 = \{\{2,3,\alpha,\overline{\alpha}\}, \{2,3,\beta,\overline{\beta}\}, \{\alpha,\overline{\alpha},\beta,\overline{\beta}\}, \{\alpha,\overline{\alpha},\overline{\beta}\}, \{2,\alpha,\beta,\overline{\beta}\}, \{2,\alpha,\overline{\alpha},\overline{\beta}\}, \{2,\beta,\overline{\alpha},\overline{\beta}\}, \{2,\overline{\alpha},\beta,\overline{\beta}\}\}$$

is the set of faces of dimension three and graphically represented by solid tetrahedra.

$$F_4 = \{2,\alpha,\overline{\alpha},\beta,\overline{\beta}\}$$

is the single face of dimension four.
The facets of $S(108)$ are

\[\{2, 3, \alpha, \overline{\alpha}\}, \{2, 3, \beta, \overline{\beta}\}, \text{ and } \{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}.\]

Note that the face $\{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ cannot be graphically represented as it is 4−dimensional.

In Section 3.1 we give a new construction in which we will be able to graphically represent a simplicial complex. Thus representing all the factorizations of 108 in $\mathbb{Z}[\sqrt{-5}]$. In particular, we return to this example in Example 3.1.3.

The next few results are taken from [BH13] but in the more general context of atomic monoids. In Section 3.2 we see similar results for the compressed irreducible divisor simplicial complex.

**Proposition 2.3.4.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Then $k_1(S(x)) = G(x)$.

*Proof.* Let $G(x) = (V, E)$ and let $k_1(S(x)) = (V', F)$. Note that $V = V'$. By definition, $F$ consists of all faces of dimension one or zero. Suppose $\{y_1, y_2\} \in E$. Then $y_1y_2 \mid x$, and thus $\{y_1, y_2\} \in F$. Suppose now that $\{y_1, y_2\} \in F$. Then $y_1y_2 \mid x$ and thus $\{y_1, y_2\} \in E$. Therefore $k_1(S(x)) = G(x)$.

**Proposition 2.3.5.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Let $A = \{a_1, \ldots, a_n\}$ be a facet of the irreducible divisor simplicial complex $S(x)$. Then there exists a factorization of $x$ of the form

\[x = a_1^{m_1} \cdots a_n^{m_n}\]

where $m_i \geq 1$ for each $i \in \{1, \ldots, n\}$.

*Proof.* By hypothesis, $A$ is a face, so $a_1 \cdots a_n \mid x$. Suppose there exists no factorization of $x$ of the form

\[x = a_1^{m_1} \cdots a_n^{m_n}\]
where $m_i \geq 1$ for each $i \in \{1, \ldots, n\}$. Then there exists irreducible elements $b_1, \ldots, b_s \notin A$ such that

$$x = (a_1^{m_1} \cdots a_n^{m_n})(b_1^{r_1} \cdots b_s^{r_s})$$

where $m_i \geq 1$, $r_j \geq 1$ for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, s\}$. Then $a_1 \cdots a_n b_1 \cdots b_s \mid x$ and so $A \subsetneq B = \{a_1, \ldots, a_n, b_1, \ldots, b_s\}$. This is a contradiction since $A$ is a facet. Therefore there exists a factorization of $x$ of the form

$$x = a_1^{m_1} \cdots a_n^{m_n}.$$

Example 2.3.2 illustrates this proposition. There are three facets, $\{2, 3\}$, $\{2, \beta, \beta\}$, and $\{3, \alpha, \alpha\}$, each of which corresponds to a factorization of $x$:

$$x = 2 \cdot 3^2, x = 2 \beta \bar{\beta}, \text{ and } x = 3\alpha \bar{\alpha}.$$

The converse of Proposition 2.3.5 is generally not true. Consider Example 2.3.3. A factorization of 108 in $\mathbb{Z}[(\sqrt{-5})^\bullet]$ is $3(\alpha)^2(\bar{\alpha})^2$, but $\{3, \alpha, \bar{\alpha}\}$ is not a facet in $S(108)$ since $\{3, \alpha, \bar{\alpha}\} \subsetneq \{2, 3, \alpha, \bar{\alpha}\}$, another face in $S(108)$. However, the following proposition gives a scenario in which the converse holds.

**Proposition 2.3.6.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$ be square-free. Then every factorization of $x$ corresponds to a facet of $S(x)$.

**Proof.** Suppose there exists a factorization of $x$ of the form

$$x = a_1 \cdots a_n$$
where each $a_i$ is an irreducible element corresponding to the face $A = \{a_1, \ldots, a_n\}$ for all $i \in \{1, \ldots, n\}$. Since $A$ is a face, there exists a facet in $S(x)$, say $B$, such that $A \subset B$, where $B = \{a_1, \ldots, a_m, b_1, \ldots, b_n\}$ where each $b_j$ is an irreducible element. By Proposition 2.3.5 and since $x$ is square-free, there exists a factorization of $x$ of the form

$$x = (a_1 \cdots a_n)(b_1 \cdots b_m).$$

Setting these factorizations of $x$ equal to each other, we obtain,

$$a_1 \cdots a_n = (a_1 \cdots a_n)(b_1 \cdots b_m)$$

By cancellation in $H$,

$$1 = b_1 \cdots b_m.$$

This is a contradiction, since each $b_j$ is irreducible in $H$. Thus $A = B$ and $A$ is a facet of $S(x)$.

Before stating the next proposition, we recall that if $X$ is a set, $\mathcal{P}(X)$ is the power set of $X$ consisting of all subsets of $X$. We write $(X, \mathcal{P}(X))$ to denote the simplicial complex with vertex set $X$ and face set $\mathcal{P}(X)$.

**Proposition 2.3.7.** Let $H$ be an atomic monoid. The following are equivalent.

1. $H$ is free.

2. For all $x \in H \setminus \{1\}$, $S(x) = (A, \mathcal{P}(A))$ for some $A \subseteq \text{Irr}(x)$. 


Proof. (1) $\Rightarrow$ (2) Suppose $H$ is free. By definition, any $x \in H \setminus \{1\}$ will factor uniquely as $x = a_1^{n_1} \cdots a_m^{n_m}$ where each $a_i$ is an irreducible element and $n_i \geq 1$ for all $i \in \{1, \ldots, m\}$. Then for any subset $B = \{a_{i_1}, \ldots, a_{i_l}\} \subseteq \{a_1, \ldots, a_n\}$, we have $B \subseteq A$ and thus $a_{i_1} \cdots a_{i_l} \mid x$. Thus $F(S(x)) = P(\{a_1, \ldots, a_n\})$. Hence, $S(x) = (A, P(A))$ for some $A \subseteq \text{Irr}(x)$. 

(2) $\Rightarrow$ (1) Let $x \in H \setminus \{1\}$. By Proposition 2.3.4, $k_1(A, P(A)) = G(x)$, where $G(x)$ is a complete graph. Since this is true for all $x \in H \setminus \{1\}$, by Theorem 2.2.5, $H$ is free.

Lastly, we wish to discuss elasticity of an element in terms of the irreducible divisor simplicial complex. As previously mentioned, Proposition 2.3.8 will improve the bound on the elasticity of an element $x$ over that given in Proposition 2.1.5.

**Proposition 2.3.8.** Let $x$ be a nonirreducible nonunit in a BFM $H$. Choose a facet $A = \{a_1, \ldots, a_t\}$ in $S(x)$ with $t + l$ maximal where $l$ is the number of loops on the $t$ vertices in $A$. Then

$$\rho(x) \leq \frac{t + l}{2}.$$ 

**Proof.** Since $x$ is a nonirreducible element, $\min L(x) \geq 2$. Now we must show that $\max L(x) \leq t + l$. Let $M = \max L(x)$. Then there exists a factorization of $x$ given by

$$x = b_1^{n_1} \cdots b_s^{n_s}$$

where each $b_i$ is an irreducible element, $b_i \neq b_j$ unless $i = j$, and $\sum_{i=1}^{s} n_i = M$. We know that $b_1 \cdots b_s \mid x$ and thus, by definition, $B = \{b_1, \ldots, b_s\}$ is a face of $S(x)$. Thus there exists a facet $C$ of $S(x)$ such that $B \subseteq C$. Then $C$ has cardinality at least $s$ and at least $l_s = \sum_{i=1}^{s} (n_i - 1)$ loops. Therefore, taking the maximum of all $s + l_s$ where $s$ is the number of vertices in a
given facet and $l_s$ is the number of loops on these vertices, we have $\max L(x) = t + l$. Hence

$$\rho(x) \leq \frac{t + l}{2}.$$  

In [BH13], it was also shown that if $x$ is square-free, then the elasticity of $x$ can be calculated exactly. The following proposition shows how this can be done in the setting of bounded factorization monoids.

**Proposition 2.3.9.** Let $x$ be a nonirreducible nonunit in a BFM $H$ where $x$ is square-free. Suppose $A \in F(S(x))$ has maximum cardinality and $B \in F(S(x))$ has minimum cardinality. Then

$$\rho(x) = \frac{\dim(A) + 1}{\dim(B) + 1}.$$  

**Proof.** By definition, $\rho(x) = \frac{\max L(x)}{\min L(x)}$. By Proposition 2.3.6, since $x$ is square-free, every factorization of $x$ corresponds to a facet in $S(x)$. In particular, a factorization of length $\max L(x)$ corresponds to the cardinality of a facet with maximum cardinality, namely $A$. Similarly, the factorization of length $\min L(x)$ corresponds to the cardinality of a facet with minimum cardinality, for example $B$. Note that the cardinality of $A$ is $\dim(A) + 1$, and the cardinality of $B$ is $\dim(B) + 1$. Therefore

$$\rho(x) = \frac{\max L(x)}{\min L(x)} = \frac{\dim(A) + 1}{\dim(B) + 1}.$$  

Further improvements on bounds for elasticity, even if $x$ is not square-free, will be given in Proposition 3.3.1, Corollary 3.3.2, and Proposition 3.3.4.
Chapter 3

Development

In this chapter we introduce the concept of compressed irreducible divisor simplicial complexes. We give results analogous of those from Chapter 2 in terms of this new construction. Also, we compare the irreducible divisor graph, compressed irreducible divisor graph, irreducible divisor simplicial complex, and the compressed irreducible divisor simplicial complex in order to gather meaningful information about the factorizations of elements in an atomic monoid. Lastly, we give improvements on the elasticity results from Chapter 2 by considering the compressed irreducible divisor simplicial complex.

3.1 Compressed Irreducible Divisor Simplicial Complexes

In this section we formally define the compressed irreducible divisor simplicial complex and give some examples of this new construction. The following definition for the compressed irreducible divisor simplicial complex of an element in an atomic monoid is analogous to how the compressed irreducible divisor graph and the irreducible divisor simplicial complex were defined.
Definition 3.1.1. Let $H$ be an atomic monoid and let $x \in H\setminus\{1\}$. The **compressed irreducible divisor simplicial complex** of $x$, denoted $S_c(x)$, is given by $(V,F)$, where

$$V = \{[y]_x : y \in \text{Irr}(H) \text{ and } y \mid x\}$$

is the vertex set of $S_c(x)$ and with $\{[y_1]_x, \ldots, [y_n]_x\} \in F$, the face set of $S_c(x)$, if and only if $y_1 \cdots y_n \mid x$.

When $x$ is clear from the context, we write $[a]$ in place of $[a]_x$.

If $f \in F(S(x))$ consists of the vertices $\{a_1, \ldots, a_n\}$, we define $[f]_x \in F(S_c(x))$ to be

$$[f]_x = \{[a_1]_x, \ldots, [a_n]_x\}$$

where we note that it may be the case that $[a_i] = [a_j]$ for $i \neq j$.

We graphically represent the compressed irreducible divisor simplicial complex in a way similar to how we graphically represented the compressed irreducible divisor graph and the irreducible divisor simplicial complex. The equivalence classes of the irreducible divisors of an element $x$ represent the vertices in the compressed irreducible divisor simplicial complex. Faces of dimension one are represented by line segments. Note that loops are not well-defined and thus we omit them just as in the compressed irreducible divisor graph. We graphically represent two-dimensional faces by shading in the corresponding triangle created by three irreducible elements whose product divides $x$ and three-dimensional faces by solid tetrahedra just as we did with irreducible divisor simplicial complexes. It is not possible to graphically represent faces of dimension higher than three so these are left to the reader’s imagination.

To illustrate this definition consider the following example.
Example 3.1.2. Let $H = \mathbb{Z}[(\sqrt{-5})^*]$ and let $x = 18$. Recall from Example 2.1.2 the only factorizations of 18 are

$$18 = 2 \cdot 3^2 = 2\beta\overline{\beta} = 3\alpha\overline{\alpha}.$$ 

We see that $\alpha$ and $\overline{\alpha}$ always appear together and that $\beta$ and $\overline{\beta}$ always appear together in factorizations of 18. Thus $[\alpha] = [\overline{\alpha}]$ and $[\beta] = [\overline{\beta}]$. Hence $S_c(18)$ is given in Figure 3.1.

![Figure 3.1: $S_c(18)$ in $\mathbb{Z}[(\sqrt{-5})^*$](image)

Notice that $S_c(18)$ in $\mathbb{Z}[(\sqrt{-5})^*$ is isomorphic to $G_c(18)$ since the compressed irreducible divisor simplicial complex has no faces of dimension more than one. This phenomena will be explained in Corollary 3.2.4.

The next example demonstrates how the compressed irreducible divisor simplicial complex of an element $x$ can be graphically represented with all faces of dimension two or less even while the irreducible divisor simplicial complex of $x$ has faces of higher dimensions, thus making it easier to see factorizations.

Example 3.1.3. Let $H = \mathbb{Z}[(\sqrt{-5})^*]$ and let $x = 108$. Recall that we are unable to graphically represent faces of dimension higher than three in $S(108)$ and in particular the facet $\{2, \beta, \overline{\beta}, \alpha, \overline{\alpha}\}$ in Example 2.3.3. However, in $S_c(108)$ we now can graphically represent this facet. The complex for $S_c(108)$ is given in Figure 3.2.
With $S_c(108) = (V, F)$, we have that

$$V = \{[2], [3], [\alpha], [\beta]\} \quad \text{and} \quad F = \emptyset \cup V \cup F_1 \cup F_2$$

where

$$F_1 = \{\{[2], [3]\}, \{[2], [\beta]\}, \{[2], [\alpha]\}, \{[3], [\alpha]\}, \{[3], [\beta]\}, \{[\alpha], [\beta]\}\}$$

and

$$F_2 = \{\{[2], [3], [\alpha]\}, \{[2], [3], [\beta]\}, \{[2], [\alpha], [\beta]\}\}.$$

We make some simple observations about the simplicial complex in Figure 3.2.

1. $F_1$ is the edge set of $G_c(108)$.

2. The sets $[2]$ and $[3]$ each have cardinality 1, whereas $[\alpha]$ and $[\beta]$ each have cardinality 2.

3. The facet $\{2, \alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ in $S(108)$ compresses to the facet $\{[2], [\alpha], [\beta]\}$ in $S_c(x)$. 
3.2 Comparing $G(x), G_c(x), S(x), \text{ and } S_c(x)$

In this section we give several results which show connections between $G(x), G_c(x), S(x),$ and $S_c(x)$ when $x$ is an element in an atomic monoid. The first result gives a result analogous to Proposition 2.3.4 for irreducible divisor simplicial complexes and compressed irreducible divisor graphs.

Proposition 3.2.1. Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Then $k_1(S_c(x)) = G_c(x)$.

Proof. Let $G_c(x) = (V, E)$ where $V = \{[y]_x : y \in \text{Irr}(H) \text{ and } y \mid x\}$ and let $k_1(S_c(x)) = (V', F)$. By definition, $V = V'$. Note that $F$ consists only of faces of dimension 1 and 0. Suppose $\{[y_1]_x, [y_2]_x\} \in E$. Then, by definition, $y_1y_2 \mid x$, which means $\{[y_1]_x, [y_2]_x\} \in F$. Suppose now that $\{[y_1]_x, [y_2]_x\} \in F$. Then $y_1y_2 \mid x$ which means $\{[y_1]_x, [y_2]_x\} \in E$. Therefore $k_1(S_c(x)) = G_c(x)$.

Using this result, we have the following two corollaries which give sufficient conditions for when $S_c(x) = G_c(x)$.

Remark 3.2.2. In Corollaries 3.2.3 and 3.2.4 when we write $S_c(x) = G_c(x)$ we mean that the simplicial complex has no faces of dimension higher than one.

We note that if $k_1(S_c(x)) = S_c(x)$, then $S_c(x) = G_c(x)$ and we have the following corollary.

Corollary 3.2.3. If $k_1(S_c(x)) = S_c(x)$, then $S_c(x) = G_c(x)$.

Corollary 3.2.4. Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Suppose $S(x) = (V, F)$ and that the following holds: If for all $f \in F$ where $\dim [f] > 2$, we have either $\dim [f] = 1$ or $\dim [f] = 0$ in $S_c(x)$, then $S_c(x) = G_c(x)$. In particular, $S_c(x)$ is a graph.
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Proof. By hypothesis, every face in $S_c(x)$ has either dimension 1 or 0. Thus $k_1(S_c(x)) = S_c(x)$. By Corollary 3.2.3, $S_c(x) = G_c(x)$.

Recall $S_c(18)$ from Example 3.1. Clearly the hypothesis of Corollary 3.2.4 holds and so $G_c(18) = S_c(18)$ as we observed earlier.

Recall Propositions 2.3.5 and 2.3.6 in Section 2.3. The following results in Propositions 3.2.5 and 3.2.6 give analogous results in terms of $S_c(x)$.

**Proposition 3.2.5.** Let $x$ be a nonirreducible nonunit in an atomic monoid $H$. Let $A = \{[a_1], \ldots, [a_n]\}$ be a facet of the compressed irreducible divisor simplicial complex $S_c(x)$. Then there exists a factorization of $x$ of the form

$$x = \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha^{m(\alpha)}$$

where each $m(\alpha)$ is a nonnegative integer depending on $\alpha$ and $x$.

Proof. By hypothesis, $A$ is a face of $S_c(x)$, so

$$\left( \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha \right) \mid x.$$ 

Suppose there exists no factorization of $x$ of the form

$$x = \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha^{m(\alpha)}$$

with each $m(\alpha) \geq 1$. Then there exist irreducible elements $\{[b_1], \ldots, [b_l]\} \not\in A$ such that

$$x = \left( \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha^{m(\alpha)} \right) \left( \prod_{j=1}^{l} \prod_{\beta \in [b_j]} \beta^{k(\beta)} \right).$$
where \( m(\alpha) \geq 1 \) and \( k(\beta) \geq 1 \). Then
\[
\left( \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha \right) \left( \prod_{j=1}^{l} \prod_{\beta \in [b_j]} \beta \right) | x
\]
and so \( A \subsetneq B \). This is a contradiction since \( A \) is a facet. Therefore there exists a factorization of \( x \) of the form
\[
x = \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha^{m(\alpha)}.
\]

Proposition 3.2.6. Let \( x \) be a nonirreducible nonunit in an atomic monoid \( H \) where \( x \) is square-free. Then every factorization of \( x \) corresponds to a facet of \( S_c(x) \).

Proof. Suppose there exists a factorization of \( x \) given by
\[
x = \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha
\]
with each \( a_i \) being an irreducible element. Note that since \( x \) is square-free, any factorization of \( x \) has no exponents on its irreducible divisors. Then \( A = \{ [a_1], \ldots, [a_n] \} \) is a face in \( S_c(x) \).

Recall that every face is contained in some facet. Thus there exists a \( B \), a facet in \( S_c(x) \), such that \( A \subsetneq B \), where \( B = \{ [a_1], \ldots, [a_n], [b_1], \ldots, [b_l] \} \). By Proposition 3.2.5, there exists a factorization of \( x \) of the form
\[
x = \left( \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha \right) \left( \prod_{j=1}^{l} \prod_{\beta \in [b_j]} \beta \right).
\]
Setting these factorizations of \( x \) equal to each other, we obtain
\[
\prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha = \left( \prod_{i=1}^{n} \prod_{\alpha \in [a_i]} \alpha \right) \left( \prod_{j=1}^{l} \prod_{\beta \in [b_j]} \beta \right).
\]
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Using the cancellation property of $H$

$$1 = \prod_{j=1}^{l} \prod_{\beta \in [b_j]} \beta,$$

a contradiction since each $b_j$ is an irreducible element. Thus $A$ is a facet.

We now give a result which uses the constructions of both $S(x)$ and $S_c(x)$ to obtain information about factorizations of an element $x$ in an atomic monoid.

**Proposition 3.2.7.** Let $H$ be an atomic monoid and let $x \in H \setminus \{1\}$. Furthermore, let $A = \{a_1, \ldots, a_t\}$ be a facet of $S(x)$ where $[a_1] = \cdots = [a_t]$ in $S_c(x)$. Then every factorization of $x$ which involves any $a_i \in A$ has the form

$$x = a_1^{n_1} \cdots a_t^{n_t},$$

for some natural numbers $n_1, \ldots, n_t$.

**Proof.** By Proposition 2.3.5 in Section 2.3, since $A$ is a facet of $S(x)$, there exists a factorization of $x$ of the form

$$x = a_1^{n_1} \cdots a_t^{n_t}$$

where $n_i \geq 1$ for each $i \in \{1, \ldots, n\}$. Since $[a_1] = \cdots = [a_t]$ in $S_c(x)$, every factorization of $x$ that contains any $a_i$ must contain all of the other $a_i$. Suppose, for the sake of contradiction, there exists factorization containing $\{a_1, \ldots, a_t\}$ but also elements $\{b_1, \ldots, b_s\}$ where $b_j \neq a_i$ for all $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$, that is

$$x = a_1^{m_1} \cdots a_t^{m_t} b_1^{k_1} \cdots b_s^{k_s},$$
for some natural numbers \( m_1, \ldots, m_t, k_1, \ldots, k_s \). Letting \( B = \{a_1, \ldots, a_t, b_1, \ldots, b_s\} \). We see that \( A \subseteq B \), but this contradicts the fact that \( A \) is a facet. Therefore every factorization of \( x \) containing any \( a_i \) has the form
\[
x = a_1^{n_1} \cdots a_t^{n_t}
\]
for some natural numbers \( n_1, \ldots, n_t \).

Along the lines of the results given in Theorem 2.1.4 from Section 2.1, Theorem 2.2.5 and Corollary 2.2.6 from Section 2.2, and Proposition 2.3.7 from Section 2.3, we now give a necessary and sufficient condition for an atomic monoid to be free in terms of the compressed irreducible divisor simplicial complex.

**Theorem 3.2.8.** Let \( H \) be an atomic monoid. The following are equivalent.

1. \( H \) is free.
2. For all \( x \in H \setminus \{1\} \), \( V(S_c(x)) = \{[a]\} \) where \( a \in \text{Irr}(x) \).

**Proof.** Suppose \( H \) is free. Then for each \( x \in H \setminus \{1\} \), \( x \) factors uniquely as
\[
x = a_1^{m_1} \cdots a_n^{m_n}
\]
where \( \text{Irr}(x) = \{a_1, \ldots, a_n\} \) and \( m_i \geq 1 \) for all \( \{1, \ldots, n\} \). Since this is the unique factorization of \( x \), clearly \( a_i \sim_x a_j \) for all \( i, j \in \{1, \ldots, n\} \). Thus, by definition, \( S_c(x) = (V, F) \) where \( F = \{[a]\} \) and \( \{a_1, \ldots, a_n\} = [a] \). Conversely, suppose \( V(S_c(x)) = \{[a]\} \) for all \( x \in H \setminus \{1\} \). Then \( G_c(x) = k_1(S_c(x)) \cong K_1 \). As this holds for all \( x \in H \setminus \{1\} \), by Theorem 2.2.5, \( H \) is free.

We conclude this section by giving two propositions that provide sufficient conditions for when \( S(x) = S_c(x) \) and \( S(x) = G(x) \).
Remark 3.2.9. In Proposition 3.2.10 we state that under certain hypotheses \( S(x) = S_c(x) \). Technically, this is not correct since the vertices of \( S(x) \) are the irreducible divisors of \( x \) and the vertices of \( S_c(x) \) are sets of irreducible divisors of \( x \). However, when there is no compression, meaning \( [a]_x = \{a\} \) for all \( a \in \text{Irr}(x) \), we consider the natural graph isomorphism mapping \( V(S(x)) \) to \( V(S_c(x)) \) by \( a \to \{a\} \) as an equivalence and write \( S(x) = S_c(x) \).

Proposition 3.2.10. Let \( H \) be an atomic monoid and let \( x \in H \setminus \{1\} \). Suppose that \( S(x) = (V, F) \), where \( F \neq \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \) for some \( a, b \in V \). Further suppose that there is no \( f \in F \), with \( |f| \geq 3 \). If \( k_1(S(x)) \) is connected, then \( S(x) = S_c(x) \).

Proof. Write \( S_c(x) = (V', F') \). First assume \( V = \{v\} \). Then \( F = \{\emptyset, \{v\}\} \). Since \( S(x) \) has only one vertex, \( [v] = v \). Thus \( V' = \{[v]\} \) and \( F' = \{\emptyset, \{[v]\}\} \). Therefore \( S(x) = S_c(x) \).

Now assume \( V \) contains at least three vertices. For the sake of contradiction, suppose \( S(x) \neq S_c(x) \). Then there exist distinct \( a, b \in V \) such that \([a]_x = [b]_x \). Since \( V \) contains at least three vertices and \( k_1(S(x)) \) is connected, there exists \( c \in V(S(x)) \) where \( c \neq a \) and \( c \neq b \) such that either \( \{a, c\} \in F(S(x)) \) or \( \{b, c\} \in F \). Without loss of generality, assume \( \{a, c\} \in F \). Since \([a]_x = [b]_x \) and \([a]_x \neq [b]_x \), every factorization containing \( a \) must also contain \( b \) and vice versa. Thus \( \{a, b, c\} \in F \), contradicting the hypothesis that \( F \) contains no face of cardinality 3. Therefore \( S(x) = S_c(x) \).

\( \square \)

Remark 3.2.11. In Proposition 3.2.12 when we write \( S(x) = G(x) \) we mean that the simplicial complex has no faces of dimension higher than one.

Proposition 3.2.12. Let \( H \) be an atomic monoid and let \( x \in H \setminus \{1\} \). Suppose that whenever \( G(x) \) contains a subgraph isomorphic to \( K_n \) for some \( n \geq 3 \) with vertices \( \{y_1, \ldots, y_n\} \) then \( y_1 \cdots y_n \nmid x \). Then \( S(x) = G(x) \).

Proof. Write \( G(x) = (V, E) \) and \( S(x) = (V, F) \), where \( V = \{y \in \text{Irr}(H) : y \mid x\} \). Suppose that \( E \neq F \). Then there exists a face in \( F \) that has dimension three or greater. Thus
there exists a complete subgraph in \( G(x) \) isomorphic to \( K_n \) for some \( n \geq 3 \) with vertex set \( \{y_1, \ldots, y_n\} \). Then \( \{y_1, \ldots, y_n\} \in F \) and \( y_1 \cdots y_n \mid x \). This contradicts the hypothesis and therefore \( S(x) = G(x) \).

\[ \Box \]

### 3.3 Elasticity

In this section we give several new results about elasticities of elements in atomic monoids utilizing the construction of the compressed irreducible divisor simplicial complex. In this first result we improve the bound on \( \min L(x) \) given certain characteristics of \( S_c(x) \).

**Proposition 3.3.1.** Let \( x \) be a nonirreducible nonunit in a BFM \( H \). Set \( m = \min \{ ||a|_x| : [a]_x \text{ is a vertex in } S_c(x) \} \). Choose a facet \( A = \{a_1, \ldots, a_t\} \) in \( S(x) \) with \( t + l \) maximal where \( l \) is the number of loops on the \( t \) vertices in \( A \). Then

\[ \rho(x) \leq \frac{t + l}{m}. \]

**Proof.** Note that \( \rho(x) = \frac{\max L(x)}{\min L(x)} \). By Proposition 2.3.8, \( \max L(x) \leq t + l \). To prove the claimed bound, we only need to show that \( m \leq \min L(x) \). Let \( x \in H \backslash \{1\} \). Then there exists a factorization of \( x \) given by

\[ x = b_1^{n_1} \cdots b_s^{n_s} \]

where each \( b_i \) is an irreducible element, \( b_i \neq b_j \) unless \( i = j \), and \( \sum_{i=1}^{s} n_i = \min L(x) \). By definition of \( S_c(x) \), for each \( i \in \{1, \ldots, s\} \), \( b_i \) is an element of \( [b]_x \) some vertex of \( S_c(x) \). Since \( m \) is the minimum size of the equivalence classes of the form \( [a]_x \), whenever an irreducible divisor occurs in a factorization of \( x \), so do at least \( m - 1 \) other elements. Thus every factorization of \( x \) involves at least \( m \) elements. Therefore \( m \leq \min L(x) \).

\[ \Box \]
An immediate Corollary to Proposition 3.3.1 is given next.

**Corollary 3.3.2.** Let $x$ be a nonirreducible nonunit in a BFM $H$. Write $S_e(x) = (V,F)$. Suppose $F = \{\emptyset, \{a_1\}, \ldots, \{a_t\}\}$. Assume that the facets of $S(x)$ are precisely the sets created by $[a_1], \ldots, [a_t]$, with a total of $l_i$ loops on the vertices $a_{i1}, \ldots, a_{im_i} \in [a_i]$. Choose $j_1 \in \{1, \ldots, t\}$ with $|[a_{j_1}]| + l_{j_1} = M$ maximal, where $l_{j_i}$ is the number of loops on the vertices $a_{j_1}, \ldots, a_{j_{l_j}}$ and $j_2 \in \{1, \ldots, t\}$ with $|[a_{j_2}]| = m$ minimal. Then

$$\rho(x) \leq \frac{M}{m}.$$

**Proof.** By hypothesis, the facets of $S(x)$ are precisely the sets created by $[a_1], \ldots, [a_t]$, with a total of $l_i$ loops on the vertices $a_{i1}, \ldots, a_{im_i} \in [a_i]$. For $j \in \{1, \ldots, t\}$, $|[a_j]|$ is the number of vertices in each facet $[a_j]$ in $S(x)$. By Proposition 3.2.5, $x$ has a factorization of the form

$$x = \prod_{\alpha \in [a_{j_1}]} \alpha^{m(\alpha)}$$

with $\sum_{\alpha \in [a_{j_1}]} m(\alpha) = t + l$. Since each of the $\alpha \in [a_{j_1}]$ occur only in factorizations with all other $\beta \in [a_{j_1}]$, $\max L(x) = t + l$. By Proposition 3.3.1,

$$\rho(x) \leq \frac{M}{m}.$$

**Remark 3.3.3.** If $m \geq 3$, then Proposition 3.3.1 and Corollary 3.3.2 give an improvement on the bound on elasticity over the bound from Proposition 2.3.8 in Section 2.3. If $m = 2$, Proposition 3.3.1 and Corollary 3.3.2 give the same bound on elasticity as in Proposition 2.3.8. If $m = 1$, Proposition 3.3.1 and Corollary 3.3.2 give a worse bound on elasticity.
In Proposition 3.3.1 and Corollary 3.3.2 we gave improvements to the known bounds for \( \min L(x) \). We now want to improve the known bound for \( \max L(x) \).

Recall the result of Proposition 2.3.8. Even when considering the compressed irreducible divisor simplicial complex we are generally unable to improve the bound on \( \max L(x) \). The problem is generally the occurrence of loops on the vertices. However, in certain circumstances, by considering both \( S(x) \) and \( S_c(x) \) for an element \( x \) in a BFM \( H \) we can make a slight improvement on this bound. This result is given in Proposition 3.3.4.

**Proposition 3.3.4.** Let \( x \) be a nonirreducible nonunit in a BFM \( H \). Choose a facet \( A = \{a_1, \ldots, a_k, a_{k+1}, \ldots, a_t\} \) in \( S(x) \) with \( t + l \) maximal where \( l \) is the number of loops on the \( t \) vertices in \( A \). Further assume that \( [a_1] = \cdots = [a_k], [a_{k+1}] = \cdots = [a_t] \), and \( [a_1] \neq [a_t] \). Also assume that for all \( j \in \{1, \ldots, t\} \), if \( \{a_j, b\} \in F(S(x)) \), then \( b = a_i \) for some \( i \in \{1, \ldots, j-1, j+1, \ldots, t\} \). Then

\[
\max L(x) \leq t + l - 1.
\]

Moreover,

\[
\rho(x) \leq \frac{t + l - 1}{m}
\]

where \( m = \min\{|[a]_x| : [a]_x \text{ is a vertex in } S_c(x)\} \).

**Proof.** By Proposition 2.3.5, since \( A \) is a facet, there exists a factorization of \( x \) given by

\[
x = a_1^{n_1} \cdots a_k^{n_k} \cdot a_{k+1}^{n_{k+1}} \cdots a_t^{n_t}
\]

where \( n_i \geq 1 \) for all \( i \in \{1, \ldots, t\} \). Since \( [a_1] = \cdots = [a_k] \neq [a_{k+1}] = \cdots = [a_t] \), then there must exist another factorization of \( x \) involving each of the vertices \( a_1, \ldots, a_k \), but none of the vertices \( a_{k+1}, \ldots, a_t \). Moreover, since there is no vertex \( b \notin \{a_1, \ldots, a_t\} \) with
\{b, a_j\} \in F(S(x)) \text{ for any } j \in \{1, \ldots, t\}, \text{ this factorization has the form}

\[x = a_1^{r_1} \cdots a_k^{r_k}\]

where \(r_s \geq 1\), for all \(s \in \{1, \ldots, k\}\). Setting these two factorizations equal to each other, we obtain

\[a_1^{n_1} \cdots a_k^{n_k} \cdot a_{k+1}^{n_{k+1}} \cdots a_t^{n_t} = a_1^{r_1} \cdots a_k^{r_k}.

For the sake of contradiction, suppose \(n_s \geq r_s\) for all \(s \in \{1, \cdots, k\}\). Then, since \(\{a_1, \ldots, a_k\} \subseteq \{a_1, \ldots, a_t\}\), we can use the cancellation property of \(H\) to obtain

\[a_1^{n_1-r_1} \cdots a_k^{n_k-r_k} a_{k+1}^{n_{k+1}} \cdots a_t^{n_t} = 1.

This is a contradiction since each \(a_i\) is an irreducible element in \(H\). Thus \(n_s < r_s\) for some \(s \in \{1, \ldots, k\}\). Therefore at least one of the loops on the vertices \(A\) does not correspond to the factorization of \(x\) of maximal length. Thus

\[\max L(x) \leq t + l - 1.\]

By Proposition 3.3.1 and the above result,

\[\rho(x) \leq \frac{t + l - 1}{m}.

\]
Example 3.3.5. Let \( H = \mathcal{B}(\mathbb{Z}/4\mathbb{Z}) \) and let \( x = 1^83^8 \). Then \( x \) factors only as:

\[
x = (1^4)(3^4)(13)^4 = (1^4)^2(3^4)^2 = (13)^8.
\]

Figures 3.3 and 3.4 show the irreducible divisor simplicial complex and the compressed irreducible divisor simplicial complex of the element \( 1^83^8 \) in \( \mathcal{B}(\mathbb{Z}/4\mathbb{Z}) \).

Note that \( \max L(x) = 8 \). By Proposition 2.3.8, \( \max L(x) \leq 3+9 = 12 \) and by Proposition 3.3.4, \( \max L(x) \leq 3+9-1 = 11 \). Although neither bound is tight, the bound in Proposition 3.3.4 does give an improvement over the bound in Proposition 2.3.8.
The goal of this chapter is to give a characterization of prime elements in an atomic monoid in terms of irreducible divisor graphs.

We first give a simple lemma from [GHK06, Proposition 1.1.8.3].

**Lemma 4.0.1.** An irreducible element $p$ in an atomic monoid $H$ is prime if and only if whenever $p$ appears in a factorization of an element $x$, it appears in every factorization of $x$.

**Proof.** Suppose $p$ is prime. Let $x \in H \backslash \{1\}$ where $p$ appears in some factorization of $x$, say

$$x = p\pi_1 \cdots \pi_m,$$

where each $\pi_i$ is an irreducible element. Suppose there exists another factorization of $x$ that does not contain $p$, for instance,

$$x = \alpha_1 \cdots \alpha_n$$

where $\alpha_j$ is irreducible and $\alpha_j \neq p$ for each $j \in \{1, \ldots, n\}$. 
Setting these factorizations of $x$ equal to each other, we obtain

$$p\pi_1 \cdots \pi_m = \alpha_1 \cdots \alpha_n.$$ 

Note that $p \mid \alpha_1 \cdots \alpha_n$ and $p$ is prime. Thus $p \mid \alpha_i$ for some $i \in \{1, \ldots, n\}$. Furthermore, since each $\alpha_j$ and $p$ are irreducible elements, $p = \alpha_j$ for some $j \in \{1, \ldots, n\}$. Therefore $p$ appears in every factorization of $x$.

Conversely, suppose that whenever $p$ occurs in a factorization of some element in $H \setminus \{1\}$, it occurs in every factorization of this element. Now suppose $p \mid ab$ for some $a, b \in H \setminus \{1\}$. Then $ab$ can be written as a product of irreducible elements, say

$$ab = a_1 \cdots a_mb_1 \cdots b_n,$$

where $a = a_1 \cdots a_m$ and $b = b_1 \cdots b_n$. By hypothesis, $p$ appears in every factorization of $ab$. Thus, since $p, a_i$, and $b_j$ are irreducible elements for each $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$, either $p = a_i$ for some $i \in \{1, \ldots, m\}$ or $p = b_j$ for some $j \in \{1, \ldots, n\}$. Thus $p \mid a$ or $p \mid b$. Therefore $p$ is prime.

In the following theorem we see that if $p$ is a prime element in an atomic monoid $H$, then $p$ is the center of a star subgraph of $G(x)$ whenever $x \in H$ and $p \mid x$.

**Theorem 4.0.2.** Let $H$ be an atomic monoid and let $p$ be an irreducible element in $H$. The following are equivalent.

1. $p$ is prime.

2. If $p \in V(G(x))$ for some $x \in H \setminus \{1\}$ with $p \neq x$, then $\{p, b\} \in E(G(x))$ for all $b \in V(G(x)) \setminus \{p\}$. 


Proof. (1) ⇒ (2) Suppose \( p \) is prime and \( p \in V(G(x)) \) for some \( x \in H \setminus \{1\} \). Then by definition, \( p \) appears in a factorization of \( x \). Since \( p \) is prime, by Lemma 4.0.1, \( p \) appears in every factorization of \( x \). Thus \( pb \mid x \) and hence \( \{p, b\} \in E(G(x)) \) for all \( b \in V(G(x)) \setminus \{p\} \).

(2) ⇒ (1) Assume 2 holds. Define

\[
S = \{ x \in H \setminus \{1\} : p \mid x \text{ but } p \text{ is not in every factorization of } x \}
\]

and let

\[
m = \min_{z \in S} \{ k : z = \pi_1 \cdots \pi_k \text{ where each } \pi_i \notin \text{Irr}(x) \setminus \{p\} \}.
\]

Then there exists \( y \in S \) such that \( y = \pi_1 \cdots \pi_m \) where each \( \pi_i \) is irreducible and no \( \pi_i \) is equal to \( p \). Note \( m \geq 2 \) since \( y \) is not irreducible. Since \( y \in S \) and \( \{p, b\} \in E(G(x)) \) for all \( b \in V(G(x)) \setminus \{p\} \), there exists another factorization of \( y \) given by \( y = p\pi_1\alpha_1 \cdots \alpha_l \), where each \( \alpha_j \) is an irreducible element. Setting the factorizations of \( y \) equal to each other, we obtain the following:

\[
y = \pi_1 \cdots \pi_m = p\pi_1\alpha_1 \cdots \alpha_l.
\]

Dividing through by \( \pi_1 \) we get,

\[
\frac{y}{\pi_1} = \pi_2 \cdots \pi_m = p\alpha_1 \cdots \alpha_l.
\]

Note that \( \frac{y}{\pi_1} \in S \) has a factorization having a length of \( m - 1 \), but this contradicts the minimality of \( m \). Hence \( S = \emptyset \) and thus every factorization of \( x \) contains \( p \). Since this holds for all \( x \in H \setminus \{1\} \), by Lemma 4.0.1, \( p \) is prime.
We conclude with the following corollary, which gives the results of Theorem 4.0.2 in terms of the irreducible divisor simplicial complex.

**Corollary 4.0.3.** Let $H$ be an atomic monoid and let $p$ be an irreducible element in $H$.

1. If $p$ is prime, then for all $x \in H \setminus \{1\}$ where $p \mid x$ and $p \neq x$, $p \in f$ for all facets $f \in F(S(x))$.

2. Suppose that for all $x \in H \setminus \{1\}$ with $\{p\} \in F(S(x))$ and $p \nmid x$, we have $\{p, b\} \in F(S(x))$ for all $b \in V(S(x)) \setminus \{p\}$. Then $p$ is prime.

**Proof.** (1) If $p$ is prime, then by Theorem 4.0.2, $\{p, b\} \in F(S(x))$ for all $b \in V(S(x)) \setminus \{p\}$. Moreover, $S(x)$ has no facet of dimension 0 since $p$ is prime, $p \mid x$, and $p \neq x$. Since every face is contained in some facet, $p \in f$ for all facets $f \in F(S(x))$.

(2) By hypothesis, part (2) of Theorem 4.0.2 holds and therefore $p$ is prime.

\qed
Bibliography


